Leverage and risk relativity: how to beat an index

HANS-PETER BERMIN AND MAGNUS HOLM

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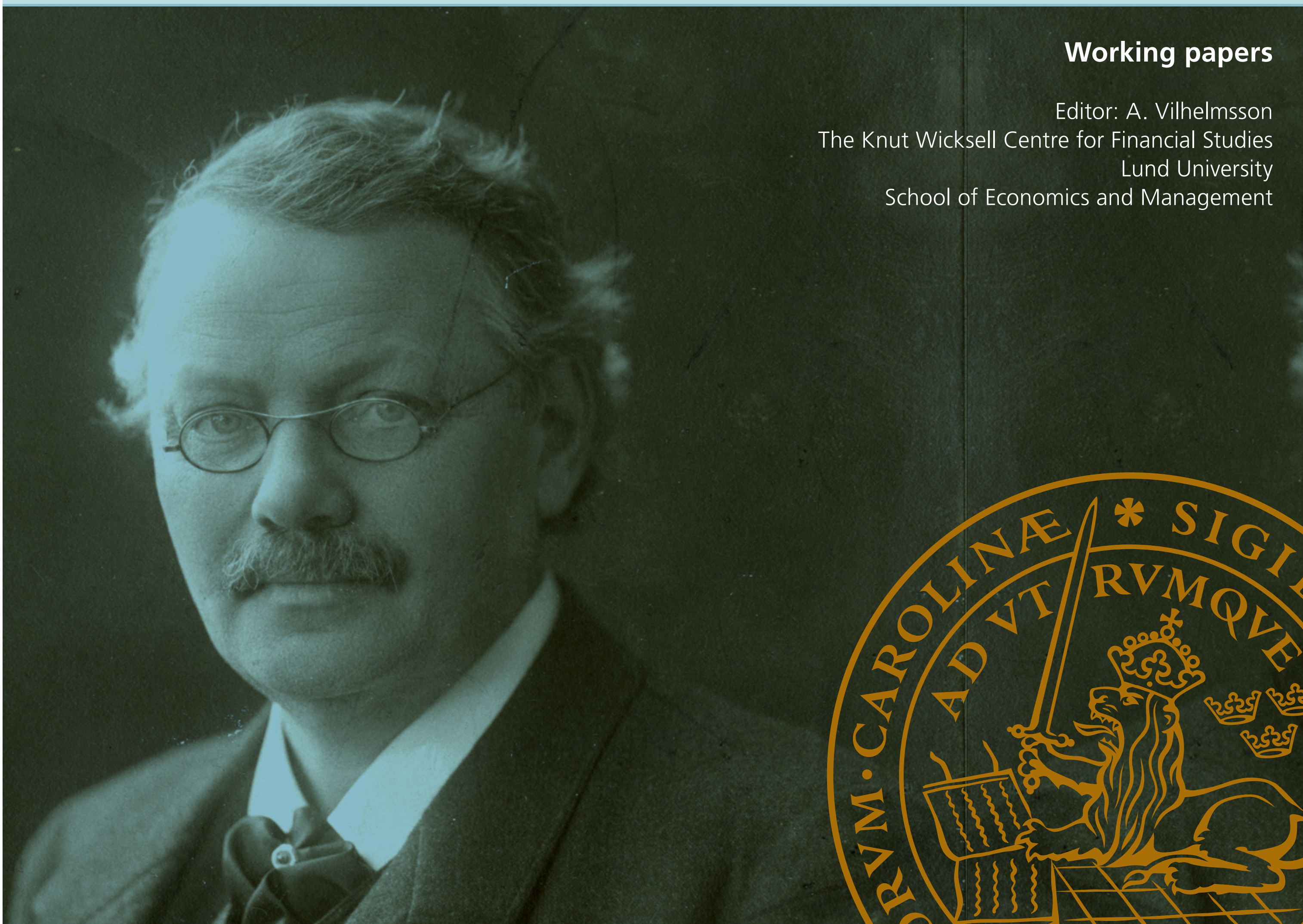

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Leverage and Risk Relativity: How to Beat an Index

A Preprint

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Abstract

In this paper we show that risk associated with leverage is fundamentally relative to an arbitrary choice of reference asset or portfolio. We characterize leverage risk as a drawdown risk measure relative to the chosen reference asset. We further prove that the growth optimal Kelly portfolio is the only portfolio for which the relative drawdown risk is not dependent on the choice of the reference asset. Additionally, we show how to translate an investor's viewpoint from one choice of reference asset to another and establish conditions for when two investors can be said to face identical leverage risk. We also prove that, for a given reference asset, the correlation between two arbitrary portfolios with identical leverage risk equals the ratio of their Sharpe ratios if and only if the leverage risk is consistently traded. More surprisingly, we observe that leverage applied to the growth optimal Kelly strategy affects the drawdown risk in much the same way as the speed of light affects velocities in Einstein's theory of special relativity. Finally, we provide details on how to trade in order to beat an arbitrary index for a given leverage risk target.

Keywords Leverage · Drawdown risk · Generalized Kelly strategy · Numéraire invariance · Risk relativity

1 Introduction

In the classical one-period mean-variance model of Markowitz [15] the portfolio allocation is fully described by the expected return and the volatility of the portfolio. When a risk-free asset is available for investment the efficient frontier consists of those trading strategies for which the Sharpe ratio is maximal. Hence, for such trading strategies there is a linear relationship between the expected excess return and the volatility. Any point on the efficient frontier can be reached by appropriately leveraging the position. This follows since, given a particular mean-variance efficient portfolio, we can construct a new allocation along the efficient frontier by simply borrowing money and investing the surplus in the existing mean-variance efficient portfolio. The larger the short position taken in the risk-free asset, the larger the expected return of the new mean-variance efficient portfolio and the larger the volatility.

While the ability to leverage provides flexibility to the market it has the tendency to generate volatile positions. Over the last two decades there has been an amplification in the usage of leverage inducing non-neglectable stress to the overall financial system. This has lead a number of researchers to pay close attention to borrowing and leverage constraints, see e.g. [4, 9, 23], in order to asses the particular risks associated with too high leverage such as: cost of margin calls, forced liquidation, losses exceeding invested capital and ultimately the risk of bankruptcy. However, the general understanding of investors attitude towards leverage is rather limited and there is a lack of consensus in how to measure this particular risk. For instance, it may seem reasonable to argue that given two portfolios, with equal expected return and volatility, a rational investor would prefer the portfolio with the lower leverage exposure. By adopting such an axiom Jacobs and Levy [7, 8] augmented the utility function of the associated mean-variance formulation with a specific term designed to capture the investor's leverage aversion. The authors showed that the introduction of a leverage aversion term (in addition to the more standard risk aversion term) resulted in lower levels of leverage than what was seen in the original

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mean-variance framework where individuals are assumed to have no aversion to leverage. Markowitz [17] replied to the criticism raised by Jacobs and Levy stating that while the original mean-variance framework in [15] largely ignore leverage risk the extended portfolio optimization framework in [16] has the potential to be modified such as to address these specific topics. It is our view, however, that none of the approaches mentioned above manage to fully explain the intrinsic nature of leverage. In general, we argue that it is questionable to associate leverage with utility and, in particular, we stress that there is no real justification in blindly believing that an investor would prefer lower leverage to higher for a given level of expected return or variance. As pointed out by Asness et al. [2], an investor can increase his expected return in two ways: either by applying leverage or by concentrating the portfolio allocation to high-risk assets. Each approach typically generates increased volatility but also delivers very different risk profiles. The point made in [2] is that leverage risk is easier than concentration risk to manage. The authors further propose a two-step allocation approach: first find the best unleveraged allocation according to some risk related criteria and second apply leverage to this portfolio to meet the desired expected return. Hence, the approach of Asness et al. is very different from that of Levy and Jacobs when addressing the risk associated with leverage.

In this paper we aim to analyze the nature of leverage and the associated risk in greater details. In order to do so we first claim that it is necessary to consider a multi-period portfolio optimization framework as opposed to the classical one-period framework. The reason being that we cannot simply superposition one-period optimal allocations if there is a positive probability of the losses exceeding the initial capital within each period. In other words, we implicitly assume that bankruptcy is an absorbing state in the sense that once an investor goes bankrupt he stays bankrupt. To facilitate the reading we present our results in a continuous-time framework since this allows us to more easily deal with the notion of bankruptcy via the concept of drawdown risk. Key to our result is the standpoint that risk in general, and drawdown risk in particular, is always relative to a chosen reference asset or portfolio. Inspired by [24], we introduce a simple yet powerful definition of relative drawdown aversion as being proportional to the logarithmic portfolio return in excess of the reference asset divided by the portfolio variance. We argue that this definition captures the essential properties associated with aversion to leverage. First, an investor who chooses not to invest in risky assets has maximal relative drawdown aversion. Second, when applying leverage to an existing portfolio the relative drawdown aversion decreases. The interpretation is that only investors with low relative drawdown aversion will apply high leverage. Third, when sufficiently high leverage is applied to an existing portfolio the relative drawdown aversion becomes negative indicating that the investor is leverage loving rather than leverage averse. Having quantified the notion of drawdown aversion we thereafter define the concept of relative drawdown risk as the reciprocal of the relative drawdown aversion. While our definition of relative drawdown risk shares many of the properties of a financial coherent or convex risk measure [1, 6], it is fundamentally different in the sense that it is not expressed in currency units. Instead it behaves more like a unitless index from which the expected maximal drawdown and the corresponding probability distribution can easily be calculated. Hence, we obtain a powerful parameterization of the leverage risk without having to adopt a utility representation of the investors.

Our main contribution is to build a comprehensive framework that allows an investor to analyze leverage risk for arbitrary trading strategies and for arbitrary reference assets (or portfolios). The objective for doing so is to answer the outstanding question: how to beat an index. The framework relies on two fundamental extensions of the Kelly theory that dates back to Kelly [11] and Latané [12]. First, and foremost, we allow for an arbitrary reference asset. We show that the growth optimal Kelly strategy is the only trading strategy for which the relative drawdown risk is not dependent on the choice of the reference asset. This observation supports the claim in [20] that the growth optimal Kelly portfolio can be used as numéraire together with the real-world probability measure for actuarial and derivative pricing. Second, we present a Kelly-like theory for arbitrary trading strategies. This allows us to identify trading strategies with similar relative drawdown risk. It also allows us to translate an investor's viewpoint from one choice of reference asset to another. There are many additional aspects which can easily be studied within the framework. For instance, we show that, for a given reference asset, the correlation between two arbitrary portfolios with identical relative drawdown risk equals the ratio of their Sharpe ratios if and only if the relative drawdown risk is traded consistently. This observation supports a claim in [3] where such a result is derived, albeit through very different methods, as a trading equilibrium. More amusingly, we find that leverage applied to the growth optimal Kelly strategy affects the relative drawdown risk in much the same way as the speed of light affects velocities in Einstein's theory of special relativity. Finally, we show that an investor trying to beat an index should always invest a fraction of his wealth in the growth optimal Kelly strategy and the remaining wealth in the index. The particular fraction chosen to invest in the growth optimal Kelly strategy depends on the drawdown risk relative to the index that the investor targets. The fact that such a simple linear trading rule is locally efficient is quite remarkable.

2 Aversion and Risk of Drawdowns

Two of the key metrics in measuring the performance of a fund manager are the realized Sharpe ratio and the drawdown. In this section we attempt to quantify the notions of drawdown aversion and drawdown risk. We take the approach that a fund manager wants the ability to allocate funds in such a way as to keep these terms constant over time, in which case they should provide easily accessible representation of the drawdown risk intrinsic to the portfolio. Thereafter, we generalize the concepts of drawdown aversion and drawdown risk by allowing the fund manager to incorporate local beliefs into his trading strategy.

We consider a capital market consisting of a number of primary assets $(P_0, P_1, \ldots, P_N)$ expressed in some common numéraire unit, say dollar. An asset related to a dividend paying stock is seen as a fund with the dividends re-invested. All assets are assumed to be positive adapted continuous processes living on a filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$, where $\mathbb{F} = \{\mathcal{F}(t) : t \geq 0\}$ is a right-continuous increasing family of σ-algebras such that $\mathcal{F}(0)$ contains all the $\mathbb{P}$-null sets of $\mathcal{F}$. As usual we think of the filtration $\mathbb{F}$ as the carrier of information. We further let P_0 be the numéraire asset of the economy, describing how the value of the numéraire unit changes over time. The rate of logarithmic return for each risky asset is denoted by

$$\mu_n(t) = \lim_{\epsilon \to 0} \frac{1}{\epsilon} \mathbb{E}\left[\log \frac{P_n(t+\epsilon)}{P_n(t)} |\mathcal{F}(t)\right], \quad n = 1, \ldots, N, \tag{1}$$

while we let r_0 denote the rate of logarithmic return for the numéraire asset. In order to remove the dependency on the numéraire unit we introduce the relative prices $P_{0|n} = P_n/P_0$ and the corresponding normalized capital market $(1, P_{0|1}, \ldots, P_{0|N})$ such that the numéraire asset can be regarded as the risk-free asset in the normalized economy. Alternatively, we can view P_0 as the new numéraire unit of the original capital market. In either case

$$\mu_{0|n}(t) = \mu_n(t) - r_0(t), \quad n = 1, \ldots, N, \tag{2}$$

equals the rate of logarithmic return corresponding to $P_{0|n}$. We also introduce the instantaneous covariance matrix of the numéraire based assets

$$V_{0|n,m}(t) = \lim_{\epsilon \to 0} \frac{1}{\epsilon} \mathbb{C}\left[\log \frac{P_{0|n}(t+\epsilon)}{P_{0|n}(t)}, \log \frac{P_{0|m}(t+\epsilon)}{P_{0|m}(t)} |\mathcal{F}(t)\right], \quad n, m = 1, \ldots, N, \tag{3}$$

and note that this matrix can conveniently be expressed in terms of the quadratic covariation process, see [10], according to

$$V_{0|n,m}(t) = \frac{d}{dt}[\log P_{0|n}, \log P_{0|m}](t), \quad n, m = 1, \ldots, N. \tag{4}$$

For the purpose of this paper we always assume that V_0 is an a.s. positive definite matrix. A standard result in linear algebra then states that V_0 generates an inner product of the form $\langle u, v\rangle_{V_0} = u' V_0 v$. To facilitate the reading we further introduce the short-hand notation

$$\sigma^2_{0|n}(t) = V_{0|n,n}(t), \quad n = 1, \ldots, N, \tag{5}$$

when referencing the instantaneous variance of the individual numéraire based assets.

Remark 2.1. In many applications the numéraire process is assumed to be locally risk-free in the sense that

$$d \log P_0(t) = r_0(t)dt.$$

In this case it is custom to identify r_0 with a continuously compounded interest rate process r such that the numéraire can be interpreted as a bank account. We also note that when the numéraire process is locally risk-free the instantaneous asset-asset covariance matrix V_0 is independent of P_0. This motivates us to simply write V and σ when dealing with examples for which this assumption is explicitly made.

From time to time, we also make references to the instantaneous rate of return vector process $b_0 = (b_{0|1}, \ldots, b_{0|n})'$, defined by

$$b_{0|n}(t) = \lim_{\epsilon \to 0} \frac{1}{\epsilon} \mathbb{E}\left[\frac{P_{0|n}(t+\epsilon) - P_{0|n}(t)}{P_{0|n}(t)} |\mathcal{F}(t)\right], \quad n = 1, \ldots, N. \tag{6}$$

A straightforward application of Itô's lemma then verifies that the rate of return relates to the rate of logarithmic return according to

$$b_{0|n}(t) = \mu_{0|n}(t) + \frac{1}{2}\sigma^2_{0|n}(t), \quad n = 1, \ldots, N. \tag{7}$$

An investor can trade in the assets and throughout this paper we assume that there are no transaction fees, that short-selling is allowed, that trading takes place continuously in time, and that the investor's trading activity does not impact the asset prices. We define a trading strategy as an $\mathbb{F}$-predictable vector process $w = (w_1, \dots, w_N)'$, representing the proportion of wealth invested in each asset, and let X_w denote the corresponding portfolio. In order to analyze the performance of the numéraire based wealth process $X_{0|w} = X_w/P_0$ we impose the restriction that, when re-balancing the portfolio, money can neither be injected nor withdrawn. Such trading strategies are said to be self-financing and satisfy

$$\frac{dX_{0|w}(t)}{X_{0|w}(t)} = \sum_{n=1}^{N} w_n(t) \frac{dP_{0|n}(t)}{P_{0|n}(t)}. \tag{8}$$

It now follows, via Itô's lemma, that if the numéraire based portfolio is almost everywhere positive the logarithmic value evolves according to

$$d \log X_{0|w}(t) = \sum_{n=1}^{N} w_n(t) \left(d \log P_{0|n}(t) + \frac{1}{2}\sigma^2_{0|n}(t)dt \right) - \frac{1}{2}\sigma^2_{0|w}(t)dt, \tag{9}$$

where the volatility $\sigma_{0|w}$ of the numéraire based portfolio satisfies

$$\sigma^2_{0|w}(t) = V_{0|w,w}(t), \quad V_{0|u,w}(t) = \frac{d}{dt}[\log X_{0|u}, \log X_{0|w}](t) = \langle u, w \rangle_{V_0}(t). \tag{10}$$

We further observe that the instantaneous rate of logarithmic return of the numéraire based portfolio takes the form

$$\mu_{0|w}(t) = \sum_{n=1}^{N} w_n(t) \left(\mu_{0|n}(t) + \frac{1}{2}\sigma^2_{0|n}(t) \right) - \frac{1}{2}\sigma^2_{0|w}(t). \tag{11}$$

Having locally characterized the investor's portfolio we proceed by examining the annualized rate of logarithmic return (or yield for short) that an investor can achieve using trading strategies for which the portfolio process is positive almost everywhere. Since, all the assets in the capital market are assumed to be continuous processes it follows from the martingale representation theorem [10] that the yield $y_{0|w}$ can be expressed as

$$y_{0|w}(T) = \frac{1}{T} \log \frac{X_{0|w}(T)}{X_{0|w}(0)} = \frac{1}{T} \int_0^T \mu_{0|w}(t)dt + \frac{1}{T} \int_0^T \sigma_{0|w}(t)dW(t), \tag{12}$$

for some Brownian motion W. By further introducing the time averages

$$\bar{\mu}_{0|w}(T) = \frac{1}{T} \int_0^T \mu_{0|w}(t)dt, \quad \bar{\sigma}^2_{0|w}(T) = \frac{1}{T} \int_0^T \sigma^2_{0|w}(t)dt, \tag{13}$$

the first two moments related to the yield takes the form

$$\mathbb{E}[y_{0|w}(T) - \bar{\mu}_{0|w}(T)] = 0, \quad \mathbb{V}[y_{0|w}(T) - \bar{\mu}_{0|w}(T)] = \frac{1}{T}\mathbb{E}[\bar{\sigma}^2_{0|w}(T)]. \tag{14}$$

Hence, for short time horizons, T, the yield fluctuates heavily around the time average $\bar{\mu}_{0|w}$, while for long time horizons these fluctuations have only marginal impact. Consequently, one sees that any notion of risk will always be strongly dependent on the investment horizon T. Below we include a more precise result for the long term performance of the investor's portfolio.

Proposition 2.2. *Let w be a trading strategy such that $X_{0|w} \geq 0$ and suppose that the time averages $\bar{\mu}_{0|w}$ and $\bar{\sigma}^2_{0|w}$ are well defined in the sense that*

$$\lim_{T\to\infty} \bar{\mu}_{0|w}(T) < \infty, \quad \lim_{T\to\infty} \bar{\sigma}^2_{0|w}(T) < \infty, \quad a.s.$$

If, in addition, $\lim_{T\to\infty} T\bar{\sigma}^2_{0|w}(T) = \infty$ a.s. then

$$\lim_{T\to\infty} \left(y_{0|w}(T) - \bar{\mu}_{0|w}(T) \right) = 0 \quad a.s.$$

Proof. The proof follows from the strong law of large numbers for continuous local martingales, see [13, Section 2.6]. That is, if we set

$$M(t) = \int_0^t \sigma_{0|w}(s)dW(s), \quad [M, M](t) = \int_0^t \sigma^2_{0|w}(s)ds,$$

we have $M/[M, M] \to 0$ if $[M, M] \to \infty$ a.s., when $t \to \infty$. Finally, since

$$y_{0|w}(t) = \bar{\mu}_{0|w}(t) + \bar{\sigma}^2_{0|w}(t) \frac{M(t)}{[M, M](t)},$$

the proof concludes. □

It is important to understand that the long term performance of any trading strategy is ultimately determined by the time average of the instantaneous rate of logarithmic return $\mu_{0|w}$. Consequently, for long investment horizons it is wrong to believe, as in Markowitz's mean-variance framework, that the realized portfolio return can be explained by its expected value. This is particularly true, as pointed out in [3], for high levels of leverage in which case the expected logarithmic return can deviate substantially from the logarithm of the expected return. Below, we show that in the long run the realized logarithmic portfolio return of any buy-hold strategy is bounded from above independently of the leverage applied.

Corollary 2.3. *Let w be a buy-hold strategy in some, or all, of the primary assets. For these assets, characterized by the set $\mathcal{N}_N = \{0 \leq n \leq N | w_n(0) \neq 0\}$ where $w_0 = 1 - \sum_{n \geq 1} w_n$, we assume that their time averages $\bar{\mu}_{0|n}$ and $\bar{\sigma}^2_{0|n}$ are well defined in the sense that*

$$\lim_{T\to\infty} \bar{\mu}_{0|n}(T) < \infty, \quad \lim_{T\to\infty} \bar{\sigma}^2_{0|n}(T) < \infty, \quad a.s.$$

If, in addition, $\lim_{T\to\infty} T\bar{\sigma}^2_{0|n}(T) = \infty$ a.s., for $1 \leq n \in \mathcal{N}_N$, then

$$\lim_{T\to\infty} \left(\frac{1}{T} \log \left| \frac{X_{0|w}(T)}{X_{0|w}(0)} \right| - \max_{n\in\mathcal{N}_N} \mu_{0|n}(T) \right) \leq 0, \quad a.s.$$

Proof. For buy-hold strategies the number of units held in each asset $q_n = w_n X_{0|w}/P_{0|n}$ is constant through time. This implies that

$$w_n(t) = w_n(0) \frac{X_{0|w}(0)}{X_{0|w}(t)} \frac{P_{0|n}(t)}{P_{0|n}(0)},$$

with

$$\frac{X_{0|w}(T)}{X_{0|w}(0)} = w_0(0) + \sum_{n=1}^{N} w_n(0) \frac{P_{0|n}(T)}{P_{0|n}(0)}, \quad w_0(0) = 1 - \sum_{n=1}^{N} w_n(0).$$

It then follows from Hölders inequality, applied to the L^p-norm, that

$$\left| \frac{X_{0|w}(T)}{X_{0|w}(0)} \right|^{\frac{1}{T}} \leq \left(|w_0(0)| + \sum_{n=1}^{N} |w_n(0)| \frac{P_{0|n}(T)}{P_{0|n}(0)} \right)^{\frac{1}{T}} \leq (N+1)^{\frac{1}{T}} \max_{n\in\mathcal{N}_N} \left(\left(|w_n(0)| \frac{P_{0|n}(T)}{P_{0|n}(0)} \right)^{\frac{1}{T}} \right).$$

The proof concludes along the lines of the proof of Proposition 2.2. □

Hence, it is the ability to dynamically trade the primary assets that allows for potential long term excessive growth and not simply the ability to leverage. However, the interesting aspect occurs when these two features are combined. In order to analyze the effect of leverage in this case we apply Eqs. (10) and (11) to obtain

$$\mu_{0|kw}(t) = k\mu_{0|w}(t) - k(k-1)\frac{1}{2}\sigma^2_{0|w}(t), \quad \sigma^2_{0|kw}(t) = k^2\sigma^2_{0|w}(t), \quad k \in \mathbb{R}. \tag{15}$$

This shows that when an investor leverages a dynamic trading strategy w, by scaling up the positions with a factor $k > 1$, there are two opposing forces. On the one hand, the instantaneous rate of logarithmic return increases linearly with the leverage applied. On the other hand, the new position is more volatile which quadraticly reduces the instantaneous rate of logarithmic return. Whichever of these forces that dominate depends on the level of leverage and the local characteristic of the initial position. What makes this observation somewhat counter intuitive is the fact that the instantaneous rate of return

$$b_{0|w}(t) = \mu_{0|w}(t) + \frac{1}{2}\sigma^2_{0|w}(t) = \sum_{n=1}^{N} w_n(t) b_{0|n}(t). \tag{16}$$

scales linearly with the level of leverage; that is $b_{0|kw} = kb_{0|w}$. In fact, this is where Markowitz's mean-variance model fails: for large leverage levels the mean-variance model must retain the expected return at the expense of an increasing probability to loose arbitrary large amounts of money. As a result, the long term realized portfolio return can be several orders of magnitude smaller than the targeted expected return. This leads us to analyze the drawdowns in greater detail.

Definition 2.4. For every $\mathbb{F}$-predictable trading strategy w we define the instantaneous relative drawdown aversion by the process

$$A_{0|w}(t) = 2\frac{\mu_{0|w}(t)}{\sigma^2_{0|w}(t)}.$$

The implications of the above definition can best be appreciated by studying how leverage impacts the relative drawdown aversion. By the use of Eqs. (15) and (16) we obtain

$$A_{0|kw}(t) = \frac{A_{0|w}(t) + 1 - k}{k} = \frac{2}{k} \frac{b_{0|w}(t)}{\sigma^2_{0|w}(t)} - 1, \quad k \in \mathbb{R}. \tag{17}$$

Hence, given that the instantaneous numéraire based rate of return $b_{0|w}$ is positive it follows that an investor who chooses not to invest in the risky assets has maximal relative drawdown aversion. We also see that in this case the relative drawdown aversion is strictly decreasing with respect to the leverage. The explanation is that only those investors with sufficiently low relative drawdown aversion will choose to apply high leverage. However, as the leverage increases the relative drawdown aversion eventually becomes negative. We say that investors taking on such trading strategies are leverage loving as opposed to leverage averse. Hence, the maximal leverage a leverage averse investor can apply to a given trading strategy w equals $k = A_{0|w} + 1$. Any attempt to leverage beyond this level will make the investor leverage loving instead of leverage averse. Below, we examine in further details the class of trading strategies for which the relative drawdown aversion is held constant.

Proposition 2.5. *Given an $\mathbb{F}$-predictable trading strategy w with a.s. constant relative drawdown aversion $A_{0|w} = A$. Assume further that the volatility is a.s. uniformly bounded and that $\lim_{T\to\infty} T\bar{\sigma}^2_{0|w}(T) = \infty$, with probability one. If we define the stopping time*

$$\tau_{0|a,b} = \inf\{t \geq 0 : X_{0|w}(t)/X_{0|w}(0) \notin (a, b)\}, \quad 0 < a < 1 < b < \infty,$$

then $\tau_{0|a,b} < \infty$ a.s. with

$$\mathbb{P}\left(\frac{X_{0|w}(\tau_{0|a,b})}{X_{0|w}(0)} = a\right) = \frac{1 - b^{-A}}{a^{-A} - b^{-A}} = 1 - \mathbb{P}\left(\frac{X_{0|w}(\tau_{0|a,b})}{X_{0|w}(0)} = b\right),$$
$$\mathbb{E}[\tau_{0|a,b}\bar{\sigma}^2_{0|w}(\tau_{0|a,b})] = \frac{2}{A}\left(\log b - \frac{1 - b^{-A}}{a^{-A} - b^{-A}} \log \frac{b}{a}\right).$$

For the limit cases it follows that $\tau_{0|0,\infty} = \infty$ a.s. with

$$\mathbb{P}(\tau_{0|0,b} < \infty)\big|_{A\geq 0} = \mathbb{P}(\tau_{0|a,\infty} < \infty)\big|_{A\leq 0} = 1.$$

Proof. The proof is provided in Appendix. □

The result above shows that a trading strategy holding the relative drawdown aversion constant is non-exploding and, consequently, avoids bankruptcy in finite time. However, such strategies will reach any upper (lower) target level in finite time if the relative drawdown aversion is less (greater) or equal than zero. This observation justifies the nomenclature of drawdown averse versus drawdown loving investors depending on the sign of the relative drawdown aversion process. In fact, we can make the connection more precise if we consider boundaries of the particular form $(1/b, b)$, $b > 1$. In this case, the probabilities in Proposition 2.5 of losses (hitting the lower barrier before the upper barrier) and gains (hitting the upper barrier before the lower barrier) simplify to

$$P_{loss}(b; A) = \mathbb{P}(X_{0|w}(\tau_{0|b^{-1},b}) = X_{0|w}(0)b^{-1}) = \frac{1}{b^A + 1}, \tag{18}$$
$$P_{gain}(b; A) = \mathbb{P}(X_{0|w}(\tau_{0|b^{-1},b}) = X_{0|w}(0)b) = P_{loss}(b; -A). \tag{19}$$

Hence, if A is a drawdown averse coefficient then $-A$ represents a drawdown loving coefficient. Note further that for a constant relative drawdown averse trading strategy these probabilities are orthogonal to the financial market in the sense that they do not explicitly depend on the portfolio characteristics $(\mu_{0|w}, \sigma_{0|w})$. Consequently, we can view them as representing drawdown risk/chance rather than financial risk/chance.

Corollary 2.6. *Let the assumptions of Proposition 2.5 hold true and set $R = 1/A$. Then, for $n \geq 0$, we have*

$$\mathbb{P}\left(\inf_{0\leq t<\infty} \log \frac{X_{0|w}(t)}{X_{0|w}(0)} \leq -nR\right)\Bigg|_{R>0} = \mathbb{P}\left(\sup_{0\leq t<\infty} \log \frac{X_{0|w}(t)}{X_{0|w}(0)} \geq -nR\right)\Bigg|_{R<0} = e^{-n},$$
$$\mathbb{E}\left[\inf_{0\leq t<\infty} \log \frac{X_{0|w}(t)}{X_{0|w}(0)}\right]\Bigg|_{R>0} = \mathbb{E}\left[\sup_{0\leq t<\infty} \log \frac{X_{0|w}(t)}{X_{0|w}(0)}\right]\Bigg|_{R<0} = -R.$$

Proof. Given the probabilities the expected values follow from straightforward calculations and is thus omitted. Furthermore, since the maximum loss set can be expressed according to

$$\{\inf_{0\leq t<\infty} X_{0|w}(t) \leq kX_{0|w}(0)\} = \{\inf_{0\leq t<\tau_{0|0,\infty}} X_{0|w}(t) \leq kX_{0|w}(0)\} = \{\inf_{0\leq t<\tau_{0|k,\infty}} X_{0|w}(t) \leq kX_{0|w}(0)\},$$

we see that

$$\mathbb{P}\left(\inf_{0\leq t<\infty} X_{0|w}(t) \leq kX_{0|w}(0)\right) = \mathbb{P}\left(X_{0|w}(\tau_{0|k,\infty}) = kX_{0|w}(0)\right).$$

The proof concludes by repeating the calculations for the maximum gain set and thereafter applying the results of Proposition 2.5. □

For a constant relative drawdown aversion strategy the term $R = 1/A$ not only captures the expected maximum relative losses (gains) an investor can face, it also allows us to express the corresponding drawdown probability distributions in a highly simple fashion. This motivates us to quantify the associated relative drawdown risk as below.

Definition 2.7. For every $\mathbb{F}$-predictable trading strategy w we define the instantaneous relative drawdown risk by the process

$$R_{0|w}(t) = \frac{1}{2}\frac{\sigma^2_{0|w}(t)}{\mu_{0|w}(t)} = \frac{1}{A_{0|w}(t)}.$$

Similar to the relative drawdown aversion process we apply leverage to gain further insights about the properties of the relative drawdown risk process. Hence, given a trading strategy w it follows from Eq. (17) that the relative drawdown risk is maximal ($R_{0|kw} = \infty$) when the leverage $k = A_{0|w} + 1$ is applied. Beyond this maximal leverage level the relative drawdown risk process changes sign indicating that the investor becomes leverage loving and thus value upside chance more than downside risk. We further show that the map $k \mapsto R_{0|kw}$ is normalized, monotonically increasing and convex over the interval where the investor is leverage averse.

Proposition 2.8. *Given a trading strategy w for which the instantaneous return $b_{0|w}$ is a.s. strictly positive. Let further k_1 and k_2 be real-valued $\mathbb{F}$-predictable process such that $0 < k_1, k_2 < A_{0|w} + 1$. Then*

$$\lim_{k_1\to 0} R_{0|k_1w}(t) = 0,$$
$$k_1(t) < k_2(t) \Rightarrow R_{0|k_1w}(t) < R_{0|k_2w}(t),$$
$$R_{0|\lambda k_1 w+(1-\lambda)k_2w}(t) < \lambda R_{0|k_1w}(t) + (1-\lambda)R_{0|k_2w}(t), \quad \lambda \in [0,1].$$

Proof. It follows from Eq. (17) that both the first and the second derivative of $R_{0|kw}$, with respect to k, are strictly positive over the interval $k \in (0, A_{0|w} + 1)$. This completes the proof. □

Although the relative drawdown risk process share similarities with coherent and convex risk measures, see [1, 6], it is important to stress that our approach is not part of the axiomatic financial risk measure theory. In particular, we do not allow for risk reduction by adding cash to the trading strategy. Instead, we simply attempt to associate a notion of leverage risk with the aversion to leverage typically seen among investors. In order to better understand our notion of relative drawdown risk let us recall that financial risk is, in practise, always measured with respect to a particular date in the future. For instance, the quintessential Value-at-Risk (VaR) describes the maximum loss an investor can face, at a future date, when a given percentage of the worst possible outcomes are ignored

$$\text{VaR}_\delta(T) = \arg\min_{y\in\mathbb{R}} \mathbb{P}(X_{0|w}(0) - X_{0|w}(T) > y) \leq 1-\delta. \tag{20}$$

In practise δ is often set to 95% or 99%, while the risk horizon T varies from one day, to ten days, to up to one year depending on the specific application. Often several risk horizons are used simultaneously in order to accurately report the financial risk of an investment and often each VaR number is associated with a high degree of model uncertainty. The reasons being that: first, VaR numbers are not easily comparable across risk horizons and secondly, the loss probability can only be calculated analytically in a few very particular cases; for instance when the volatility of the portfolio is held constant and a constant relative drawdown aversion strategy is employed. We avoid (some of) these problems by specifying the risk event with a discretization in space rather than in time. In doing so, the probabilities associated with the risk events do not explicitly depend on the portfolio characteristics when a trading strategy with constant relative drawdown aversion, $A = 1/R$, is employed. For such trading strategies, a minor modification of the VaR measure yields

$$X_{0|w}(0)(1-(1-\delta)^R) = \arg\min_{y\in\mathbb{R}} \mathbb{P}\left(X_{0|w}(0) - \inf_{0\leq t<\infty} X_{0|w}(t) > y\right)\Big|_{R>0} \leq 1-\delta. \tag{21}$$

The above formula, derived using Corollary 2.6, highlights yet another difference between a financial risk measure and our proposed instantaneous relative drawdown risk process; namely that while the range of a financial risk measure corresponds to the numéraire based potential losses the relative drawdown risk behaves more like a unitless index.

3 Kelly Strategies and Leverage

In this section we extend our analysis to include the second key metric of a fund manager; that is the Sharpe ratio. In particular, we examine the draw down aversion from a Kelly trader's perspective. When the numéraire is identified with a locally risk free bank account the trading strategy $w^* = V_0^{-1}b_0$ is known as the growth optimal Kelly strategy. Bermin and Holm [3] show that such a trading strategy has a maximal instantaneous Sharpe ratio in the sense of [19]. In order to extend their results to a market with a general numéraire process we notice that the Sharpe ratio, as initially introduced in [21] and subsequently in [22], gives little advice on how to deal with a general numéraire asset. We proceed by noting that the instantaneous portfolio characteristics can be expressed in terms of the numéraire based return according to

$$b_{0|w}(t) = \lim_{\epsilon \to 0} \frac{1}{\epsilon} \mathbb{E} \left[\frac{X_{0|w}(t+\epsilon) - X_{0|w}(t)}{X_{0|w}(t)} |\mathcal{F}(t) \right], \tag{22}$$

$$\sigma^2_{0|w}(t) = \lim_{\epsilon \to 0} \frac{1}{\epsilon} \mathbb{V} \left[\frac{X_{0|w}(t+\epsilon) - X_{0|w}(t)}{X_{0|w}(t)} |\mathcal{F}(t) \right]. \tag{23}$$

These observations motivate the extension below.

Definition 3.1. Given a capital market with an arbitrary numéraire asset P_0. For every $\mathbb{F}$-predictable trading strategy w we define the instantaneous Sharpe ratio according to

$$s_{0|w}(t) = \frac{b_{0|w}(t)}{\sigma_{0|w}(t)}.$$

One notes that when the numéraire asset can be identified with a bank account, as in Eq. (2.1), our definition of instantaneous Sharpe ratio corresponds to that of [19].

Theorem 3.2 (Kelly). *Any $\mathbb{F}$-predictable trading strategy w that maximizes the magnitude of the instantaneous Sharpe ratio $s_{0|w}$ is of the form*

$$w(t) = k(t)w^*(t), \quad w^*(t) = V_0^{-1}(t)b_0(t) = \arg\max_{w} \mu_{0|w}(t),$$

for some real-valued $\mathbb{F}$-predictable process k. We call such strategies Kelly strategies and we refer to the process k as the Kelly multiplier. The instantaneous squared Sharpe ratio of a Kelly strategy is independent of k and satisfies

$$s^2_{0|w}(t) = s^2_{0|w^*}(t) = \langle b_0, b_0 \rangle_{V_0^{-1}}(t).$$

The corresponding logarithmic return and volatility of such a strategy satisfy

$$\mu_{0|w}(t) = \frac{1}{2}k(t)\left(2 - k(t)\right) s^2_{0|w^*}(t), \quad \sigma^2_{0|w}(t) = k^2(t) s^2_{0|w^*}(t),$$

such that $\mu_{0|w}$ is maximal for $k = 1$. We call w^* the growth optimal Kelly strategy.

Proof. Let us first observe that, with $w^* = V_0^1 b_0$, the rate of return process $b_{0|w}$ can be expressed according to

$$b_{0|w}(t) = w'(t)b_0(t) = \langle w, w^* \rangle_{V_0}(t).$$

Hence, the instantaneous Sharpe ratio takes the form

$$s_{0|w}(t) = \frac{b_{0|w}(t)}{\sigma_{0|w}(t)} = \frac{\langle w, w^* \rangle_{V_0}(t)}{\sqrt{\langle w, w \rangle_{V_0}(t)}}.$$

It now follows from the Cauchy-Schwartz's inequality that $s^2_{0|w} \leq \langle w^*, w^* \rangle_{V_0} = s^2_{0|w^*}$ with equality if and only if w and w^* are collinear. The local characteristics of such a trading strategy $w = kw^*$ can easily be computed and is thus omitted. For an arbitrary trading strategy, however, it follows that

$$\mu_{0|w}(t) = b_{0|w}(t) - \frac{1}{2}\sigma^2_{0|w}(t) = \langle w, w^* \rangle_{V_0}(t) - \frac{1}{2}\langle w, w \rangle_{V_0}(t).$$

The first order condition, with respect to w, then gives us $V_0(w^* - w) = 0$. Hence, $\arg\max_w \mu_{0|w} = w^*$ since V_0 is invertible. $\square$

In order to build confidence in the Kelly approach we compare the historical realised level curves of the yield $y_{0|w}$ with those of the estimated time average $\bar{\mu}_{0|w}$ as outlined in Proposition 2.2. In Fig. 1 we plot the yield of various constant trading strategies w for the the total return index (dividends re-invested) of S&P500 and for gold, using Eq. (8) with daily re-balancing over the last 15 years. We also estimate constant model parameters over this time period from which we derive an estimate of $\bar{\mu}_{0|w}$. This shows that approximately 4000 data points can, to first order, be accurately explained by a model of six parameters only. The second order effects are mainly two: first we notice that daily re-balancing is not frequent enough for highly leveraged trading strategies and secondly, we believe that additional accuracy can likely be obtained by allowing the model parameters to fluctuate around the constant means within the period. Additionally, we remind the reader that the theoretical level curves of $\bar{\mu}_{0|w}$ are based on in-sample estimates, where only the first and last data points are effectively used to estimate the logarithmic sample returns μ_1, μ_2 and r. Hence, normal care should be taken when using the model for predictions of the future.

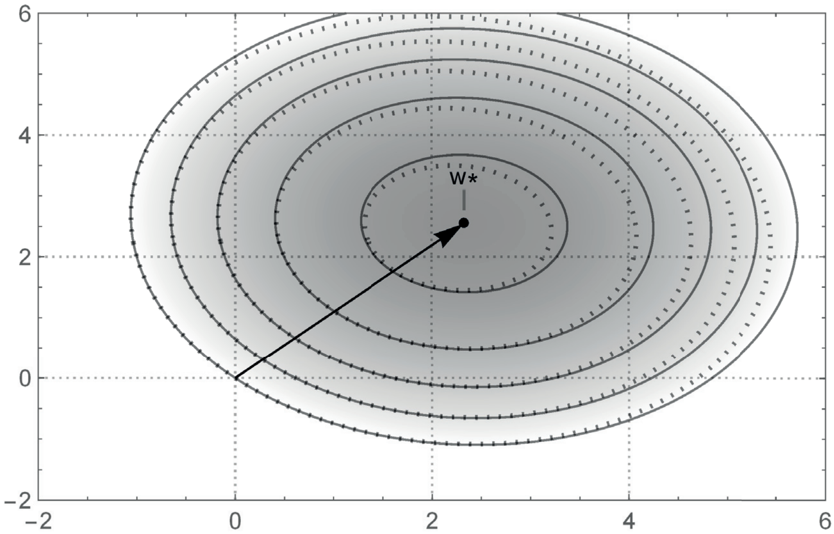

Figure 1: This figure shows the level curves of the daily traded realised yield $y_{0|w}$ (dotted lines) and the estimated time average $\bar{\mu}_{0|w}$ (solid line), for various constant trading strategies $w = (w_1, w_2)'$. We let w_1 and w_2 denote the wealth fractions held in the total return index of S&P500 (x-axis) and gold (y-axis), respectively, and consider the time period Nov. 2004 to Nov. 2020. The underlying model parameters are assumed to be constant with estimates: $\mu_1 = 0.0891$, $\sigma_1 = 0.196$, $\mu_2 = 0.0867$, $\sigma_2 = 0.183$, $\rho = 0.0394$ (source: finance.yahoo.com, tickers: SPY and GLD). We further take the numéraire to equal the US-bank account with constant interest rate $r = 0.015$ and display the implied growth optimal Kelly strategy $w^* = (2.335, 2.542)'$. It also follows from Definition 2.7 that the two assets are traded at similar levels of relative drawdown risk with $R_{0|w} = 0.259$ for S&P500 and $R_{0|w} = 0.234$ for gold.

The result presented in Theorem 3.2 is a generalization of the fractional Kelly strategies in [14]. Similar to [3] we allow the Kelly multiplier to be a real-valued stochastic process but we also allow for an arbitrary numéraire process. The latter extension is, in part, motivated by the works of Davis and Lleo [5], who show that a terminal utility maximizing investor can in some situations be characterized as investing a constant wealth fraction in the growth optimal Kelly strategy and the remaining part in a particular mutual fund related to the intertemporal hedge portfolio in Merton [18]. It follows that the instantaneous mean-variance frontier is parameterized by $k \in [0, 2]$ since any value of the Kelly multiplier outside this region generates a negative rate of instantaneous logarithmic return. Furthermore, the efficient instantaneous mean-variance frontier is characterized by $k \in [0, 1]$ since for any $k \in [1, 2]$ it is more efficient (in terms of achieving the same logarithmic mean at a lower variance) to use the Kelly multiplier $2 - k$. In other words, when funds are allocated according to the Kelly criterion it is never optimal to leverage more than the growth optimal Kelly strategy. Furthermore, the instantaneous relative drawdown aversion of a Kelly strategy takes the simple form

$$A_{0|kw^*}(t) = \mathcal{A}(k(t)), \quad \mathcal{A}(k) = \frac{2}{k} - 1. \tag{24}$$

Hence, for a mean-variance efficient Kelly strategy the instantaneous relative drawdown aversion is always greater or equal than one. Moreover, for any Kelly strategy the relative drawdown aversion is uniquely defined by the Kelly multiplier k and equals $\mathcal{A}(k)$. The representation in Eq. (24) can further be extended to arbitrary trading strategies as described below.

Definition 3.3. For every $\mathbb{F}$-predictable trading strategy w we define the relative leverage risk process

$$k_{0|w}(t) = \frac{\sigma^2_{0|w}(t)}{b_{0|w}(t)}.$$

such that the relative drawdown aversion and drawdown risk processes equal $A_{0|w} = \mathcal{A}(k_{0|w})$ and $R_{0|w} = \mathcal{A}(2-k_{0|w})$.

It is easily seen that there is a one-to-one correspondence between the processes $(k_{0|w}, A_{0|w}, R_{0|w})$, except for the points $k_{0|w} = 0$ and $k_{0|w} = 2$ where $A_{0|w}$ and $R_{0|w}$, respectively, are not defined. By excluding the latter point we see that two trading strategies have the same relative drawdown risk if their relative leverage risk processes are identical. We also see that for Kelly strategies the relative leverage risk process coincides with the Kelly multiplier as $k_{0|kw^*} = k$. In this case, the relative drawdown risk process behave quantitatively similar to the relative leverage risk process along the efficient instantaneous mean-variance frontier, $k \in [0, 1]$, but indicates a far greater risk over the inefficient mean-variance frontier, represented by $k \in (1, 2)$. For k outside of $[0, 2]$ the relative drawdown risk process is negative indicating that we are entering a drawdown loving territory. If we interpret $-R_{0|w}$ as a drawdown chance process it follows that a Kelly strategy with $k = 1 + \delta$, $\delta \geq 1$, is preferable to one for which the Kelly multiplier $k = 1 - \delta$. The reason being that, although both strategies face the same negative logarithmic drift $\mu_{0|w}$, the higher the magnitude of the Kelly multiplier the higher the volatility and consequently the higher the probability of hitting an upper barrier before a lower barrier. In other words, while both such strategies are loosing in the sense of Kelly, the strategy with the higher magnitude of the Kelly multiplier has a better upside chance [2]. Finally, we mention that it is also possible to define the relative leverage aversion process $1/k_{0|w}$ and that this process is closely related to Arrow-Pratt's definition of relative risk aversion, see [3] for details.

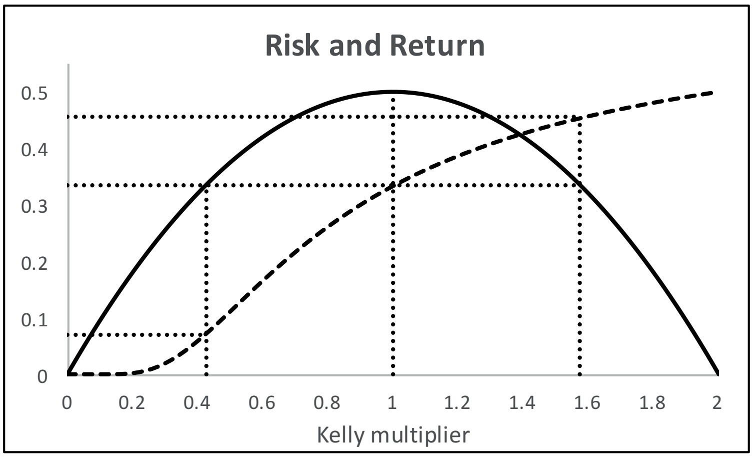

Figure 2: This plot shows $P_{loss}\,(2; \mathcal{A}(k))$ (dashed line), representing the probability of halving the numéraire based wealth before doubling, and $\Psi(k) = k(2-k)/2$ in Theorem 3.2 (solid line), representing the excess logarithmic return a Kelly investor can obtain for a unit instantaneous Sharpe ratio, as functions of the Kelly multiplier k.

It is further illuminating to study various probability distributions associated with the relative drawdown measure. Inspired by [24], we plot in Fig. 2 the probability that an investor will half his (numéraire based) wealth before doubling, as derived in Eq. (18). We also plot the function $\Psi(k) = k(2-k)/2$ in Theorem 3.2, indicating the excess return a Kelly investor can obtain in relation to the maximal instantaneous Sharpe ratio. We see that $k = 1$ maximizes Ψ, with $\Psi(1) = 1/2$, while the probability of halving the wealth before doubling equals $1/3$. This is indeed a risky position to take on. If we instead require $\Psi = 1/3$, we notice that there are two possible Kelly multipliers that meet this condition: $k = 0.42$ and $k = 1.58$. The corresponding probabilities equal 0.07 and 0.45, respectively. We also confirm that it is never efficient to use a Kelly multiplier greater than one and that an investor can considerably reduce his risk, by lowering the Kelly multiplier, without a major reduction of the excess return. The expected time to hit the boundaries, see Proposition 2.5, is: for $k = 0.42(1.00)$, and a constant instantaneous Sharpe ratio, equal to $1.79(0.46)/s^2_{0|w^*}$ years.

We also compare the results in Corollary 2.6 with market data for S&P500, gold and the US-bank account, as outlined in Fig. 1. However, in doing so we must consider a finite risk horizon. Without going into the details we state that when

[2]For readers familiar with laser physics one notices that the risk process $R_{0|w}$ behaves similar to temperature, following the increasing path $0_+ \to +\infty \to -\infty \to 0_-$. With probability one an upper (lower) barrier will be hit first if $R_{0|w} = 0_+$ ($R_{0|w} = 0_-$)

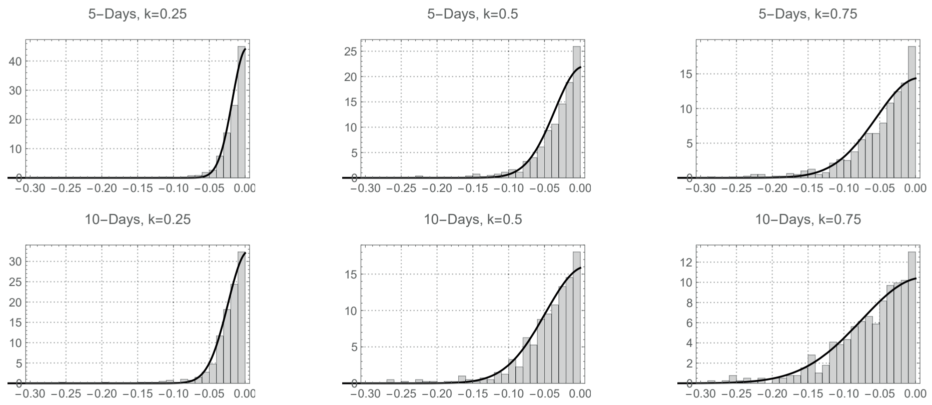

Figure 3: This figure shows the daily traded realised 5 and 10 days drawdown densities for various Kelly strategies, $k \in \{1/4, 1/2, 3/4\}$, on S&P500, gold and the US-bank account, over the interval Nov. 2004 to Nov. 2020. For comparison we also include the theoretical drawdown densities corresponding to Eq. (25). The constant model parameters used are similar to those in Fig. 1.

the initial model parameters $(r, \mu_1, \mu_2, \sigma_1, \sigma_2, \rho)$ are constant the resulting portfolio characteristics $(\mu_{0|w}, \sigma_{0|w})$ are constant as well. In this special case it follows from the reflection principle of Brownian motion that

$$\mathbb{P}\left(\inf_{0\le t<T} \log \frac{X_{0|w}(t)}{X_{0|w}(0)} \le -nR\right)\Bigg|_{n,R>0} = N\left(\frac{-nR - \mu_{0|w}T}{\sigma_{0|w}\sqrt{T}}\right) + e^{-n} N\left(\frac{-nR + \mu_{0|w}T}{\sigma_{0|w}\sqrt{T}}\right) \tag{25}$$

In Fig. 3 we plot the realised (daily traded) density function over the time period Nov. 2004 to Nov. 2020 for various Kelly strategies, $k \in \{1/4, 1/2, 3/4\}$, and two risk horizons, $T \in \{5, 10\}$ days. For comparison we also plot the theoretical density function based on estimated constant parameters over the same interval. Similar to Fig. 1, we find a good agreement between market behaviour and theory.

4 Generalized Kelly Strategies

In the previous section we showed that for Kelly strategies drawdown risk and leverage risk were fully explained by the Kelly multiplier. In this section we aim to build a similar framework for arbitrary trading strategies. The goal is to determine under what circumstances two different trading strategies can be said to have equivalent drawdown risk. We derive a generalization of the Kelly criteria formalizing the thoughts of Asness et al. [2]. That is, we show how to leverage an arbitrary trading strategy such that the drawdown risk is comparable to that of a Kelly strategy.

Theorem 4.1 (Generalized Kelly)**.** *Let v be an arbitrary trading strategy and define*

$$w(t) = k(t)\hat{v}(t), \quad \hat{v}(t) = \frac{1}{k_{0|v}(t)} v(t),$$

for some real-valued $\mathbb{F}$-predictable process k. We call such strategies w generalized Kelly strategies and we refer to the process k as the Kelly multiplier. The instantaneous squared Sharpe ratio of a generalized Kelly strategy is independent of k and satisfies

$$s^2_{0|w}(t) = s^2_{0|\hat{v}}(t) = s^2_{0|v}(t).$$

The corresponding logarithmic drift and volatility of such a strategy satisfy

$$\mu_{0|w}(t) = \frac{1}{2} k(t)\left(2 - k(t)\right) s^2_{0|v}(t), \quad \sigma^2_{0|w}(t) = k^2(t) s^2_{0|v}(t),$$

such that $\mu_{0|w}$ is maximal for $k = 1$. We call $\hat{v}$ the optimal generalized Kelly strategy and further observe that

$$k_{0|w}(t) = k(t).$$

Proof. Straightforward calculations yield

$$b_{0|w}(t) = \langle w, w^* \rangle_{V_0}(t) = k(t) b_{0|\hat{v}}(t) = \frac{k(t)}{k_{0|v}(t)} b_{0|v}(t),$$
$$\sigma^2_{0|w}(t) = \langle w, w \rangle_{V_0}(t) = k^2(t) \sigma^2_{0|\hat{v}}(t) = \frac{k^2(t)}{k^2_{0|v}(t)} \sigma^2_{0|v}(t).$$

Hence, it follows from Definition 3.1 that $s^2_{0|w} = s^2_{0|\hat{v}} = s^2_{0|v}$. Moreover, using Definition 3.3, we obtain

$$b_{0|w}(t) = k(t) s^2_{0|v}(t), \quad \sigma^2_{0|w}(t) = k^2(t) s^2_{0|v}(t),$$

from which we see that $k_{0|w} = \sigma^2_{0|w}/b_{0|w} = k$. Finally, we observe that the logarithmic drift

$$\mu_{0|w}(t) = b_{0|w}(t) - \frac{1}{2}\sigma^2_{0|w}(t) = \frac{1}{2} k(t)\left(2 - k(t)\right) s^2_{0|v}(t),$$

is maximal for $k = 1$, which completes the proof. □

We note that Theorem 4.1 reduces to Theorem 3.2 if we let the arbitrary trading strategy v equal a Kelly strategy. This follows since

$$v(t) = k(t) w^*(t) \Rightarrow \hat{v}(t) = \frac{1}{k_{0|kw^*}(t)} k(t) w^*(t) = w^*(t). \tag{26}$$

More importantly, Theorem 4.1 states that the trading strategy $\hat{v}$ has the same relative leverage risk as the growth optimal Kelly strategy w^* for any given trading strategy v. This implies that a generalized Kelly strategy and a Kelly strategy have identical relative leverage and drawdown risk if and only if they apply the same Kelly multiplier k. Additionally, it follows that the associate level curves are convex as indicated below.

Proposition 4.2. *Let $\hat{v}^1$, $\hat{v}^2$ be optimal generalized Kelly strategies and let $k \geq 0$ be a common Kelly multiplier, that is a real-valued $\mathbb{F}$-predictable process. Then*

$$k_{0|\lambda k\hat{v}^1 + (1-\lambda)k\hat{v}^2}(t) \leq \lambda k_{0|k\hat{v}^1}(t) + (1-\lambda) k_{0|k\hat{v}^2}(t), \quad \lambda \in [0,1].$$

Proof. Straightforward calculations shows that

$$\langle \hat{v}^1, \hat{v}^2 \rangle_{V_0}(t) \leq \frac{1}{2}(\sigma^2_{0|\hat{v}^1}(t) + \sigma^2_{0|\hat{v}^2}(t)),$$

since Eq. (10) implies that

$$0 \leq \sigma^2_{0|\hat{v}^1 - \hat{v}^2}(t) = \sigma^2_{0|\hat{v}^1}(t) + \sigma^2_{0|\hat{v}^2}(t) - 2\langle \hat{v}^1, \hat{v}^2 \rangle_{V_0}(t).$$

Consequently, for $\lambda \in [0,1]$, we obtain

$$\begin{aligned}\sigma^2_{0|\lambda\hat{v}^1 + (1-\lambda)\hat{v}^2}(t) &= \lambda^2 \sigma^2_{0|\hat{v}^1}(t) + (1-\lambda)^2 \sigma^2_{0|\hat{v}^2}(t) + 2\lambda(1-\lambda)\langle \hat{v}^1, \hat{v}^2 \rangle_{V_0}(t),\\ &\leq \lambda \sigma^2_{0|\hat{v}^1}(t) + (1-\lambda)\sigma^2_{0|\hat{v}^2}(t).\end{aligned}$$

Furthermore, since $b_{0|\lambda\hat{v}^1 + (1-\lambda)\hat{v}^2} = \lambda b_{0|\hat{v}^1} + (1-\lambda) b_{0|\hat{v}^2}$ and $k_{0|\hat{v}} = 1$ for any optimal generalized Kelly strategy $\hat{v}$, we see that

$$k_{0|\lambda\hat{v}^1 + (1-\lambda)\hat{v}^2}(t) = \frac{\sigma^2_{0|\lambda\hat{v}^1 + (1-\lambda)\hat{v}^2}(t)}{b_{0|\lambda\hat{v}^1 + (1-\lambda)\hat{v}^2}(t)} \leq \frac{\lambda\sigma^2_{0|\hat{v}^1}(t) + (1-\lambda)\sigma^2_{0|\hat{v}^2}(t)}{\lambda b_{0|\hat{v}^1}(t) + (1-\lambda) b_{0|\hat{v}^2}(t)} = 1.$$

Finally, since $k_{0|kw} = k k_{0|w}$ for any trading strategy and $\lambda k_{0|\hat{v}^1} + (1-\lambda) k_{0|\hat{v}^2} = 1$ the proof concludes. □

Remark 4.3. The convexity of the level curves corresponding to the relative leverage risk carry over to the level curves of the relative drawdown risk in the sense that for generalized Kelly strategies $w^1 = k\hat{v}^1$ and $w^2 = k\hat{v}^2$, where $k \in (0,2)$, we have

$$R_{0|\lambda w^1 + (1-\lambda)w^2}(t) \leq \lambda R_{0|w^1}(t) + (1-\lambda) R_{0|w^2}(t), \quad \lambda \in [0,1].$$

Note, however, that this relation might not hold true for arbitrary trading strategies w^1, w^2 and not even generalized Kelly strategies with different Kelly multipliers. Hence, care should be taken when adding this result to Proposition 2.8.

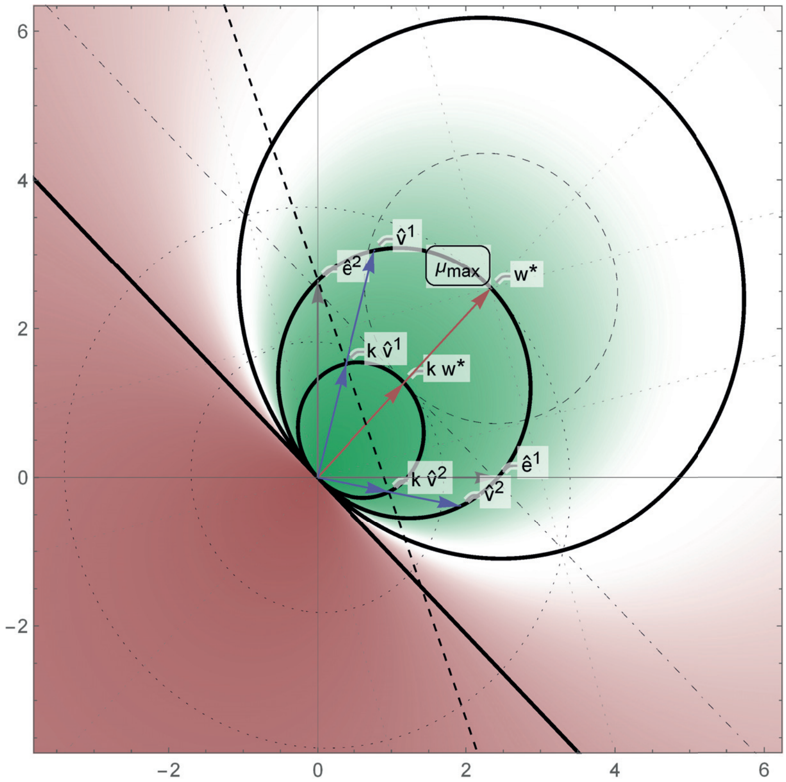

Figure 4: In this plot we show the growth optimal Kelly strategy w^* for S&P500 and gold, based on the historical estimates reported in Fig. 1, together with the optimal generalized single name Kelly strategies $\hat{e}^1$, $\hat{e}^2$ and two arbitrary optimal generalized Kelly strategies $\hat{v}^1$, $\hat{v}^2$. We highlight the level curves of the relative leverage risk for $k_{0|w} \in \{k, 1, 2, \pm\infty, -1\}$ (solid thick lines), where $k \in (0,1)$. Note that the relative leverage risk level curves for $k_{0|w} = \pm\infty$ coincide and take the form $b_{0|w} = 0$. We also show the level curves of the instantaneous variance $\sigma^2_{0|w}$ (dotted lines) and the level curves of the rate of logarithmic return $\mu_{0|w}$ (dashed lines). Note further that the level curves of $k_{0|w}$ and $\sigma^2_{0|w}$ are separated from those of $\mu_{0|w}$ by lines of the form $b_{0|w} = c$, $c \in \mathbb{R}$, (dashed-dotted line).

We also notice that the relative drawdown aversion process, for a generalized Kelly strategy, takes the familiar form

$$A_{0|k\hat{v}}(t) = \mathcal{A}(k_{0|k\hat{v}}(t)) = \mathcal{A}(k(t)), \quad \mathcal{A}(k) = \frac{2}{k} - 1. \tag{27}$$

Furthermore, due to the particular structure of the logarithmic return and the volatility it follows from Theorem 4.1 that the efficient allocations are those for which $k \in [0,1]$. In other words, the only difference between a Kelly strategy and a generalized Kelly strategy is that the Kelly strategy describes the instantaneous mean-variance allocations for which the the instantaneous Sharpe ratio is maximal, that is the mean-variance frontier.

In Fig. 4 we illustrate the risk profile for Kelly trading the indices S&P500 and gold. We use the notation

$$e^i = (\mathbf{1}\{i=1\}, \ldots, \mathbf{1}\{i=N\})', \quad i = 1, \ldots, N, \tag{28}$$

to represent the single name strategies such that $\hat{e}^i$ denotes the corresponding optimal generalized Kelly strategy. As can be seen in the figure, the plane is divided by the line $R_{0|w} = -1$ or equivalently $k_{0|w} = \pm\infty$. We call this line the Markowitz line and notice that it describes the trading strategies for which the instantaneous rate of return $b_{0|w} = 0$. As pointed out by Markowitz [15] the area above (under) this line has a positive (negative) expected rate of return. What Markowitz misses, though, is that as we increase the leverage $k > 2$ the expected logarithmic return becomes negative. This increases the probability of the portfolio $X_{0|w}$ to hit any given lower barrier as explained in Proposition 2.5. It is worth stressing, however, that the optimal trading strategies in the mean-variance model of Markowitz are of the same form as those of the Kelly strategies. The reason being, as can be seen in Fig. 4, is that the optimization problem of minimizing the instantaneous variance $\sigma^2_{0|w}$ subject to a constraint of the form $b_{0|w} = c$, $c > 0$, is identical to maximizing the instantaneous rate of logarithmic return $\mu_{0|w}$ for a fixed value of either the instantaneous variance or of the instantaneous relative leverage risk $k_{0|w}$. The power of the Kelly theory lies in the fact that $\mu_{0|w}$ attains its maximum along the Kelly line. This therefore sets an upper bound on the maximum leverage, relative to the growth optimal Kelly strategy, that can be applied. It is also evident from Fig. 4 that all the level curves are convex, which further provides a geometrical interpretation of Proposition 4.2. In higher dimensions it follows that any $\mathbb{R}^N$-valued trading strategy w can be represented as an element in $S^{N-1} \times \mathbb{R}$, that is we parameterize the level sets S^{N-1} by the Kelly multiplier k. Hence, the generalized Kelly strategies provide a powerful mean to identify portfolios with equivalent relative draw down risk.

5 Risk Relativity

In this section we show that risk is fundamentally relative to a choice of an arbitrary risk-free asset or portfolio. However, what makes an asset risk-free is a subjective criteria. For example, a dollar investor will consider euro a risky asset, while a euro investor will consider dollar a risky asset. Hence, in order to asses risk one must start by making a choice of what risk-free signifies. We show that the growth optimal Kelly strategy is the only trading strategy for which the associated (leverage) risk is not dependent on the choice of the risk-free asset. We also establish a numéraire invariant framework describing how to translate an investor's viewpoint from one choice of risk-free asset to another. Finally, we show that correlations and Sharpe ratios are closely interlinked when the relative drawdown risk is traded consistently.

5.1 Numéraire Invariant Framework

In order to study how an investor can compare his portfolio with a general reference portfolio we extend the numéraire invariant framework introduced earlier. We use the notation $X_{u|w} = X_w/X_u$ to refer to the ratio of portfolios using the trading strategies w and the reference strategy u, respectively. Hence, it is clear that $X_{u|w}$ is independent of the original numéraire unit and that X_u can be regarded as the risk-free asset in the normalized capital market $(P_{u|0}, P_{u|1}, \ldots, P_{u|N})$, where $P_{u|n} = P_n/X_u$. However, since none of the primary assets are no longer risk-free the self-financing condition reads

$$\frac{dX_{u|w}(t)}{X_{u|w}(t)} = \left(1 - \sum_{n=1}^{N} w_n(t)\right) \frac{dP_{u|0}(t)}{P_{u|0}(t)} + \sum_{n=1}^{N} w_n(t) \frac{dP_{u|n}(t)}{P_{u|n}(t)}. \tag{29}$$

One notes that the reference portfolio X_u can be related to the market numéraire asset P_0 by letting u equal the zero vector $\mathbf{0} = (0, \ldots, 0)'$. More precisely, it follows from the identity $X_{u|w} = X_{0|w}/X_{0|u}$ and Eq. (9) that

$$d \log X_{\mathbf{0}|w}(t) = d \log X_{0|w}(t), \tag{30}$$

which shows that $X_{\mathbf{0}}$ represents a constant scaling of P_0 and can therefore be identified with the original market numéraire. We also see that the rate of logarithmic return for the process $X_{u|w}$ can be expressed according to

$$\mu_{u|w}(t) = \mu_{0|w}(t) - \mu_{0|u}(t). \tag{31}$$

Next, let us define the instantaneous covariance process and the corresponding variance process

$$V_{u|w^1,w^2}(t) = \frac{d}{dt}[\log X_{u|w^1}, \log X_{u|w^2}](t), \quad \sigma^2_{u|w}(t) = V_{u|w,w}(t). \tag{32}$$

We further let $b_{u|w}$ denote the rate of return process and state that Itô's formula then yields

$$b_{u|w}(t) = \lim_{\epsilon \to 0} \frac{1}{\epsilon} \mathbb{E}\left[\frac{X_{u|w}(t+\epsilon) - X_{u|w}(t)}{X_{u|w}(t)} | \mathcal{F}(t)\right] = \mu_{u|w}(t) + \frac{1}{2}\sigma^2_{u|w}(t). \tag{33}$$

Definition 5.1. Given an arbitrary $\mathbb{F}$-predictable reference strategy u. For every $\mathbb{F}$-predictable trading strategy w we define the instantaneous generalized Sharpe ratio according to

$$s_{u|w}(t) = \frac{b_{u|w}(t)}{\sigma_{u|w}(t)}.$$

In order to characterize the set of trading strategies for which the instantaneous generalized Sharpe ratio is maximal, given a fixed reference strategy u, we first introduce a convenient notation after which we present an auxiliary result relating the instantaneous covariance process V_u to the original covariance process V_0.

Definition 5.2. Given an arbitrary $\mathbb{F}$-predictable reference strategy u and an $\mathbb{F}$-predictable trading strategy w we define

$$w_u(t) = w(t) - u(t).$$

Lemma 5.3. *For every reference strategy u, the instantaneous covariance process V_u and the rate of return process b_u can conveniently be expressed in terms of the covariance process V_0, related to the original market numéraire P_0, according to*

$$\begin{aligned} V_{u|w^1,w^2}(t)) &= V_{0|w_u^1,w_u^2}(t) = \langle w_u^1, w_u^2 \rangle_{V_0}(t), \\ b_{u|w}(t) &= V_{0|w_u,w_u^*}(t) = \langle w_u, w_u^* \rangle_{V_0}(t). \end{aligned}$$

Proof. Since $X_{u|w} = X_{0|w}/X_{0|u}$ it follows from Eq. (32) that

$$\begin{aligned} V_{u|w^1,w^2}(t) &= \frac{d}{dt}[\log X_{0|w^1} - \log X_{0|u}, \log X_{0|w^2} - \log X_{0|u}](t), \\ &= V_{0|w^1,w^2}(t) - V_{0|w^1,u}(t) - V_{0|w^2,u}(t) + V_{0|u,u}(t). \end{aligned}$$

Hence, by the use of Eq. (10) and Definition 5.2 we obtain

$$V_{u|w^1,w^2}(t) = \langle w^1, w^2 \rangle_{V_0}(t) - \langle w^1, u \rangle_{V_0}(t) - \langle w^2, u \rangle_{V_0}(t) + \langle u, u \rangle_{V_0}(t) = \langle w_u^1, w_u^2 \rangle_{V_0}(t).$$

For the second result we first notice, through direct calculations, that

$$\sigma^2_{0|w_u}(t) + \sigma^2_{0|u}(t) - \sigma^2_{0|w}(t) = -2\langle w_u, u \rangle_{V_0}(t).$$

We then have

$$\begin{aligned} b_{u|w}(t) &= \mu_{u|w}(t) + \frac{1}{2}\sigma^2_{u|w}(t), \\ &= \mu_{0|w}(t) - \mu_{0|u}(t) + \frac{1}{2}\sigma^2_{0|w_u}(t), \\ &= b_{0|w}(t) - \frac{1}{2}\sigma^2_{0|w}(t) - b_{0|u}(t) + \frac{1}{2}\sigma^2_{0|u}(t) + \frac{1}{2}\sigma^2_{0|w_u}(t), \\ &= b_{0|w}(t) - b_{0|u}(t) - \langle w_u, u \rangle_{V_0}(t). \end{aligned}$$

The proof concludes since $b_{0|w} - b_{0|u} = \langle w_u, w^* \rangle_{V_0}$ as demonstrated in the proof of Theorem 3.2. □

It is important to notice that while the instantaneous covariance process V_u and the rate of return process b_u depend on the reference strategy u, they can always be evaluated using the original covariance process V_0. We further see that the reference portfolio X_u indeed represents the risk-free asset since both $\mu_{u|u}$ and $\sigma_{u|u}$ are identically equal to zero. More generally, we define the risk concepts as per below.

Definition 5.4. For every $\mathbb{F}$-predictable reference strategy u and trading strategy w we define the relative drawdown risk process $R_{u|w}$ and the relative leverage risk process $k_{u|w}$ according to

$$R_{u|w}(t) = \frac{1}{2}\frac{\sigma^2_{u|w}(t)}{\mu_{u|w}(t)}, \quad k_{u|w}(t) = \frac{\sigma^2_{u|w}(t)}{b_{u|w}(t)}.$$

Note that the relative drawdown process can further be linked to a drawdown probability by a corresponding generalization of Corollary 2.6. We now have all the pieces in place to generalize Theorem 3.2.

Theorem 5.5 (Kelly Invariance)**.** *For every $\mathbb{F}$-predictable reference strategy u we have*

$$\arg\max_{w} \mu_{u|w}(t) = w^*(t).$$

Furthermore, the trading strategy w that maximizes the magnitude of the instantaneous generalized Sharpe ratio $s_{u|w}$, for any given reference strategy u, satisfies

$$w_u(t) = k(t)w_u^*(t),$$

for some real-valued $\mathbb{F}$-predictable process k. We call such strategies Kelly strategies and we refer to the process k as the Kelly multiplier. The instantaneous squared generalized Sharpe ratio of a Kelly strategy is independent of k and satisfy

$$s^2_{u|w}(t) = s^2_{u|w^*}(t).$$

The corresponding logarithmic drift and volatility of such a strategy are given by

$$\mu_{u|w}(t) = \frac{1}{2}k(t)\left(2 - k(t)\right)s^2_{u|w^*}(t), \quad \sigma^2_{u|w}(t) = k^2(t)s^2_{u|w^*}(t),$$

such that $\mu_{u|w}$ is maximal for $k = 1$. We further observe that

$$k_{u|w}(t) = k(t).$$

Proof. It follows from Lemma 5.3 that

$$\mu_{u|w}(t) = b_{u|w}(t) - \frac{1}{2}\sigma^2_{u|w}(t) = \langle w_u, w^*_u\rangle_{V_0}(t) - \frac{1}{2}\langle w_u, w_u\rangle_{V_0}(t)$$

Hence, the first order condition for w_u-optimality reads $V_0(w^*_u - w_u) = \mathbf{0}$. The first part of the proof now follows from the invertibility of V_0.

For the second part we observe, similar to Theorem 3.2, that Cauchy-Schwartz's inequality and Lemma 5.3 implies that the square of the instantaneous generalized Sharpe ratio in Definition 5.1 satisfies

$$s^2_{u|w}(t) = \frac{b^2_{u|w}(t)}{\sigma^2_{u|w}(t)} = \frac{\langle w_u, w^*_u\rangle^2_{V_0}(t)}{\langle w_u, w_u\rangle_{V_0}(t)} \leq \langle w^*_u, w^*_u\rangle_{V_0}(t) = s^2_{u|w^*}(t),$$

with equality if and only if w_u and w^*_u are collinear; that is $w_u = kw^*_u$ for some $\mathbb{F}$-predictable real-valued process k. The proof concludes by evaluating the logarithmic drift and the volatility, with a trading strategy of this form, and noting that the maximal logarithmic return is achieved at $k = 1$. □

As Theorem 5.5 shows, the growth optimal Kelly strategy w^* is invariant with respect to the reference strategy u. This is at first sight quite a surprising result since the maximal instantaneous generalized Sharpe ratio, in general, depends on both the trading strategy and the reference strategy. Furthermore, only when the growth optimal Kelly strategy is taken as the reference strategy is the instantaneous generalized Sharpe ratio independent of the trading strategy. However, in this case we have the trivial but important result

Corollary 5.6. *For every $\mathbb{F}$-predictable trading strategy w we have*

$$s_{w^*|w}(t) = 0.$$

Proof. The proof follows directly from Definition 5.1 and Lemma 5.3. □

Hence, not only is the growth optimal Kelly strategy invariant with respect to each investors reference strategy; no other trading strategy can achieve a higher generalized Sharpe ratio than what is implied from the growth optimal Kelly strategy.

Next we show that the Kelly strategies satisfy a fundamental equation of balance between logarithmic return and volatility. In order to do so we first generalize the concept of relative drawdown aversion.

Definition 5.7. For every $\mathbb{F}$-predictable reference strategy u and trading strategy w we define the instantaneous relative drawdown aversion by the processes

$$A_{u|w}(t) = 2\frac{\mu_{u|w}(t)}{\sigma^2_{u|w}(t)}.$$

Similar to Eq. (24), it follows that the relative drawdown aversion for a Kelly strategy $w = u + kw^*_u$ is invariant with respect to the reference strategy u and only depends on the Kelly multiplier as Theorem 5.5 implies that

$$A_{u|u+kw^*_u}(t) = \mathcal{A}(k(t)), \quad \mathcal{A}(k) = \frac{2}{k} - 1. \tag{34}$$

Hence, for any reference strategy u, a Kelly strategy satisfies the fundamental equation of balance between logarithmic return and volatility

$$\mu_{u|u+kw^*_u}(t) = \frac{1}{2}\mathcal{A}(k(t))\sigma^2_{u|u+kw^*_u}(t). \tag{35}$$

In order to further characterize the Kelly strategies we look at the relative drawdown and leverage risk. While these risk measures, in general, depend on both the trading strategy and the leverage strategy, for Kelly strategies the dependency is fully summarized by the Kelly multiplier

$$R_{u|u+kw^*_u}(t) = \frac{k(t)}{2 - k(t)}, \quad k_{u|u+kw^*_u}(t) = k(t). \tag{36}$$

This implies that two Kelly traders employing the same Kelly multiplier but different reference strategies will both report the same relative leverage risk when asked to characterize their positions. Yet, when one of the Kelly traders is asked to characterize both positions the relative leverage risk will be reported differently. The reason is that from the first Kelly traders perspective the second Kelly trader is not using a Kelly strategy. We further see, by letting $k \to 0$, that the risk-free asset X_u has zero drawdown risk relative to the reference strategy u. In other words, $R_{u|u} = 0$ for any choice of reference strategy.

Theorem 5.8 (Relative Drawdown Risk of Growth Optimal Kelly Strategy)**.** *For every $\mathbb{F}$-predictable reference strategy $u \neq w^*$ we have*

$$R_{u|w^*}(t) = 1.$$

Furthermore, $R_{w^*|w^*} = 0$ and $R_{w^*|w}(t) = -1$ for every trading strategy $w \neq w^*$.

Proof. The main result follows directly from Eq. (36), setting $k = 1$. We then use the relationship $R_{u|w} = -R_{w|u}$ to prove that $R_{w^*|w}(t) = -1$ for every trading strategy $w \neq w^*$. The final result follows from the observation that $R_{u|u} = 0$ for any reference strategy. □

It is interesting to observe that no matter how close the reference strategy is to the growth optimal Kelly strategy, the latter strategy always has a constant relative drawdown risk. For instance, we see that $R_{kw^*|w^*} = 1$, for any value of $k \neq 1$. This is analog to the speed of light in Special Relativity, no matter at what speed you move you still experience that the maximum speed, that of light, is constant in any reference frame. In fact, the similarities run even deeper.

Theorem 5.9 (Relative Drawdown Risk Velocity of Kelly Strategies)**.** *Given an $\mathbb{F}$-predictable reference strategy u and let w^1, w^2 be Kelly strategies of the form $w^1_u = k_1 w^*_u$ and $w^2_u = k_2 w^*_u$. Then*

$$R_{w^1|w^2} = \frac{R_{u|w^2} - R_{u|w^1}}{1 - R_{u|w^2} R_{u|w^1}}.$$

Proof. For any Kelly strategy $w = u + kw^*_u$ we know that $R_{u|w} = k/(2-k)$. Therefore

$$\frac{R_{u|w^2}(t) - R_{u|w^1}(t)}{1 - R_{u|w^2}(t) R_{u|w^1}(t)} = \frac{\frac{k_2(t)}{2-k_2(t)} - \frac{k_1(t)}{2-k_1(t)}}{1 - \frac{k_2(t)}{2-k_2(t)} \frac{k_1(t)}{2-k_1(t)}} = \frac{k_2(t) - k_1(t)}{2 - (k_2(t) + k_1(t))} = \frac{\frac{k_2(t)-k_1(t)}{1-k_1(t)}}{2 - \frac{k_2(t)-k_1(t)}{1-k_1(t)}}.$$

In general, however, we know that $R_{w^1|w^2} = k_{w^1|w^2}/(2 - k_{w^1|w^2})$. Hence, the proof follows once we proved that the relative leverage risk process satisfies

$$k_{w^1|w^2}(t) = \frac{k_2(t) - k_1(t)}{1 - k_1(t)}.$$

By the use of Lemma 5.3 and Definition 5.4 we obtain

$$k_{w^1|w^2}(t) = \frac{\langle w^2 - w^1, w^2 - w^1 \rangle_{V_0}(t)}{\langle w^2 - w^1, w^* - w^1 \rangle_{V_0}(t)} = \frac{(k_2(t) - k_1(t))^2 \langle w^*_u, w^*_u \rangle_{V_0}(t)}{(k_2(t) - k_1(t))(1 - k_1(t)) \langle w^*_u, w^*_u \rangle_{V_0}(t)},$$

which completes the proof. □

Remark 5.10. Notice how the velocity of the relative drawdown risk process, parameterized in terms of the Kelly multiplier, satisfy the same reference change transformation as velocity does in Special Relativity. In other words, consider two objects moving in the same direction. If an independent observer measures their velocities as v^1 and v^2, respectively, then an observer travelling along the first object would measure the velocity of the second object as $(v^2 - v^1)/(1 - v^2 v^1/c^2)$. Note that this expression is identical to ours if we consider velocities relative to the speed of light. Hence, while the normalizing constant in physics equals the speed of light, the normalizing trading strategy in economics equals the growth optimal Kelly strategy.

Next, we provide a curious result indicating that there always exists a particular Kelly strategy for which the relative drawdown risk is invariant with respect to the reference strategy.

Proposition 5.11. *Given two $\mathbb{F}$-predictable reference strategies u^1, u^2 and consider a trading strategy w of the form $w_{u^1} = k_{u^1|u^2} w^*_{u^1}$. Then*

$$R_{u^2|w}(t) = R_{u^1|w}(t) = R_{u^1|u^2}(t).$$

Proof. Since $w = u^1 + k_{u^1|u^2}(w^* - u^1)$ is a Kelly strategy with respect to u^1 it follows directly from Theorem 5.5 that $k_{u^1|w} = k_{u^1|u^2}$. Next, we notice that

$$k_{u^2|w}(t) = \frac{\langle w - u^2, w - u^2 \rangle_{V_0}(t)}{\langle w - u^2, w^* - u^2 \rangle_{V_0}(t)} = k_{u^1|u^2}(t) + \frac{\langle w - u^2, w - u^2 - k_{u^1|u^2}(w^* - u^2) \rangle_{V_0}(t)}{\langle w - u^2, w^* - u^2 \rangle_{V_0}(t)}.$$

Straightforward calculations now yield

$$\begin{aligned}\langle w - u^2, &w - u^2 - k_{u^1|u^2}(w^* - u^2) \rangle_{V_0}(t) = \langle u^2 - u^1 - k_{u^1|u^2}(w^* - u^1), (1 - k_{u^1|u^2})(u^2 - u^1) \rangle_{V_0}(t), \\ &= (1 - k_{u^1|u^2}) \left(\langle u^2 - u^1, u^2 - u^1 \rangle_{V_0}(t) - k_{u^1|u^2} \langle u^2 - u^1, w^* - u^1 \rangle_{V_0}(t) \right).\end{aligned}$$

Hence, by using the definition of $k_{u^1|u^2}$ this term vanishes. The proof concludes by converting the result to the relative drawdown risk. □

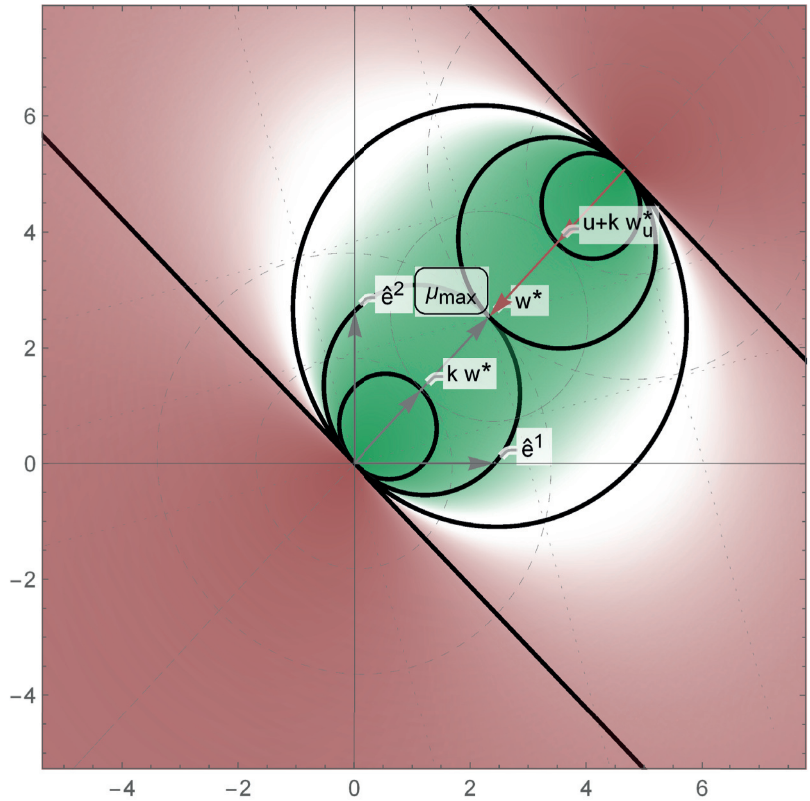

Figure 5: This picture shows the Kelly strategies $w = u + kw^*_u$ for the reference strategies $u = \mathbf{0}$ and $u = 2w^*$. We highlight the level curves of leverage risk for $k_{u|w} \in \{k, 1, 2, \pm\infty\}$ (solid thick lines), where $k \in (0, 1)$. We also show the level curves of the instantaneous variance $\sigma^2_{u|w}$ (dotted lines) and the level curves of the logarithmic return $\mu_{u|w}$ (solid thin lines).

However, in order to exemplify that risk is generally relative we show in Fig. 5 the two Kelly strategies corresponding to the reference strategies $u = \mathbf{0}$ and $u = 2w^*$. Both Kelly strategies agree on the profitable area (that is the area where the logarithmic return $\mu_{u|w} > 0$) but completely disagree on which Kelly strategy is efficient ($k \in (0, 1)$) and which is inefficient ($k \in (1, 2)$). Note further that the joint profitable area according to Markowitz is now sandwiched between the two lines $b_{\mathbf{0}|w} = 0$ and $b_{2w^*|w} = 0$.

Finally, in order to easily identify the level curves of the relative leverage risk we generalize Theorem 4.1.

Theorem 5.12 (Generalized Kelly). *Given an $\mathbb{F}$-predictable reference strategy u and let v be an arbitrary trading strategy. Define the trading strategies w and $\hat{v}$ according to*

$$w_u(t) = k(t)\hat{v}_u(t), \quad \hat{v}_u(t) = \frac{1}{k_{u|v}(t)} v_u(t),$$

for some real-valued $\mathbb{F}$-predictable process k. We call such strategies w generalized Kelly strategies and we refer to the process k as the Kelly multiplier. The instantaneous squared generalized Sharpe ratio of a generalized Kelly strategy is independent of k and satisfies

$$s^2_{u|w}(t) = s^2_{u|\hat{v}}(t) = s^2_{u|v}(t).$$

The corresponding logarithmic drift and volatility of such a strategy satisfy

$$\mu_{u|w}(t) = \frac{1}{2}k(t)\,(2 - k(t))\, s^2_{u|v}(t), \quad \sigma^2_{u|w}(t) = k^2(t) s^2_{u|v}(t),$$

such that $\mu_{u|w}$ is maximal for $k = 1$. We call $\hat{v}$ the optimal generalized Kelly strategy and further observe that

$$k_{u|w}(t) = k(t).$$

Proof. The proof follows the exact steps of Theorem 4.1, albeit with an arbitrary reference strategy, and is thus omitted. □

Corollary 5.13. *Given an $\mathbb{F}$-predictable reference strategy u and let v be a Kelly strategy such that $v_u = kw^*_u$, for some Kelly multiplier k. Then*

$$\hat{v}(t) = w^*(t).$$

Proof. Explicit calculations yield

$$\hat{v}(t) = u(t) + \frac{1}{k_{u|v}(t)}\,(v(t) - u(t)) = u(t) + \frac{k(t)}{k_{u|u+kw^*_u}(t)} w^*_u(t).$$

The proof concludes since $k_{u|u+kw^*_u} = k$ for any reference strategy u. □

Hence, with k denoting the Kelly multiplier of a generalized Kelly strategy $w = u + k\hat{v}_u$, it follows from Definition 5.4 that the relative drawdown risk process satisfies

$$R_{u|u+k\hat{v}_u}(t) = \frac{1}{A_{u|u+k\hat{v}_u}(t)} = \frac{1}{\mathcal{A}(k_{u|u+k\hat{v}_u}(t))} = \frac{1}{\mathcal{A}(k(t))}. \tag{37}$$

In particular, we see that $R_{u|\hat{v}} = 1$ for any reference strategy u. Note further that every generalized Kelly strategy satisfies the fundamental equation of balance between logarithmic return and volatility

$$\mu_{u|u+k\hat{v}_u}(t) = \frac{1}{2}\mathcal{A}(k(t))\sigma^2_{u|u+k\hat{v}_u}(t). \tag{38}$$

Similar to Proposition 4.2 we can also prove that the level curves of the relative leverage risk process $k_{u|w}$ are convex for every fixed reference strategy. Furthermore, it is not hard to show that any reference strategy u for which $k_{0|u} = 2$ has the property that the level curves of $k_{u|w} = 2$ coincide with those of $k_{0|w} = 2$. These level curves also agree with the level curves of $\mu_{u|w} = 0$ and $\mu_{0|w} = 0$; details are left to the reader.

This completes the risk picture and highlights the fact that risk is a subjective concept depending on the point of reference; thus sharply in contrast to the optimal Kelly strategy which is an objective concept independent of the reference point.

5.2 Change of Reference Frame

It is often of interest to describe how the local characteristics and the relative drawdown risk of a trading strategy depend on the reference strategy. For instance, a fund manager trading in multiple currencies might want the ability to easily translate, say, dollar-risk to euro-risk and vice versa. Below we provide the details of how this work in our framework.

In order to get started we notice that there is one evident transformation from which many of the other results follow; namely

$$\mu_{u^2|w}(t) = \mu_{u^1|w}(t) - \mu_{u^1|u^2}(t). \tag{39}$$

This result follows trivially from taking the logarithm of the identity $X_{u^2|w} = X_{u^1|w}/X_{u^1|u^2}$. We now transform this equation to the risk processes as described below

Proposition 5.14 (Partial Change of Reference - Relative Drawdown Risk)**.** *Given $\mathbb{F}$-predictable reference strategies u^1, u^2 and a trading strategy w. Then*

$$\frac{\sigma^2_{u^2|w}(t)}{R_{u^2|w}(t)} = \frac{\sigma^2_{u^1|w}(t)}{R_{u^1|w}(t)} - \frac{\sigma^2_{u^1|u^2}(t)}{R_{u^1|u^2}(t)}.$$

Proof. The proof follows directly from Definition 5.4 and Eq. (39). □

Hence, given that we know how to obtain the volatility process $\sigma_{u^2|w}$ we can evaluate the new relative drawdown risk process $R_{u^2|w}$ and consequently the new relative leverage risk process according to $k_{u^2|w} = 2R_{u^2|w}/(1 + R_{u^2|w})$.

Proposition 5.15 (Change of Reference - Covariance)**.** *Given $\mathbb{F}$-predictable reference strategies u^1, u^2 and trading strategies w^1, w^2. Then*

$$V_{u^2|w^1,w^2}(t) = V_{u^1|w^1,w^2}(t) - V_{u^1|u^2,w^1}(t) - V_{u^1|u^2,w^2}(t) + \sigma^2_{u^1|u^2}(t).$$

Proof. The proof follows from algebraic manipulations using Lemma 5.3 as described below

$$V_{u^1|w^1,w^2}(t) - V_{u^1|u^2,w^1}(t) = \langle w^1_{u^1}, w^2_{u^1}\rangle_{V_0}(t) - \langle u^2_{u^1}, w^1_{u^1}\rangle_{V_0}(t) = \langle w^1_{u^1}, w^2_{u^2}\rangle_{V_0}(t),$$

$$\sigma^2_{u^1|u^2}(t) - V_{u^1|u^2,w^2}(t) = \langle u^2_{u^1}, u^2_{u^1}\rangle_{V_0}(t) - \langle u^2_{u^1}, w^2_{u^1}\rangle_{V_0}(t) = \langle u^2_{u^1}, u^2_{w^2}\rangle_{V_0}(t) = \langle u^1_{u^2}, w^2_{u^2}\rangle_{V_0}(t).$$

Furthermore, since

$$\langle w^1_{u^1}, w^2_{u^2}\rangle_{V_0}(t) + \langle u^1_{u^2}, w^2_{u^2}\rangle_{V_0}(t) = \langle w^1_{u^2}, w^2_{u^2}\rangle_{V_0}(t) = V_{u^2|w^1,w^2}(t),$$

the proof concludes. □

We now have all the elements in place to describe the transformation of the local characteristics.

Corollary 5.16 (Change of Reference - Local Characteristics)**.** *Given $\mathbb{F}$-predictable reference strategies u^1, u^2 and a trading strategy w. Then*

$$\sigma^2_{u^2|w}(t) = \sigma^2_{u^1|w}(t) - 2V_{u^1|u^2,w}(t) + \sigma^2_{u^1|u^2}(t),$$

$$b_{u^2|w}(t) = b_{u^1|w}(t) - b_{u^1|u^2}(t) - V_{u^1|u^2,w}(t) + \sigma^2_{u^1|u^2}(t).$$

Proof. The first result follows directly from Proposition 5.15 and is thus omitted. The second result follows from expressing Eq. (39) according to

$$b_{u^2|w}(t) - \frac{1}{2}\sigma^2_{u^2|w}(t) = b_{u^1|w}(t) - \frac{1}{2}\sigma^2_{u^1|w}(t) - b_{u^1|u^2}(t) + \frac{1}{2}\sigma^2_{u^1|u^2}(t),$$

and then applying the first result. □

Note that the result above further allows us to express the transformation of the instantaneous generalized Sharpe ratio. The details are left to the reader. Instead we provide an example highlighting the difference between Kelly trading and Markowitz trading.

Example 5.17. Let us fix two reference strategies u^1, u^2 and assume, for simplicity, that $\mu_{u^1|u^2} = 0$. In this case, the rate of return processes take the form

$$b_{u^1|u^2}(t) = b_{u^2|u^1}(t) = \frac{1}{2}\sigma^2_{u^1|u^2}(t) = \frac{1}{2}\sigma^2_{u^2|u^1}(t),$$

and the corresponding relative leverage risk processes equal $k_{u^1|u^2}(t) = k_{u^2|u^1}(t) = 2$. Hence, it follows from Theorem 5.12 that the optimal generalized Kelly strategy $\hat{u}^1$, given the reference strategy u^2, and the optimal generalized Kelly strategy $\hat{u}^2$, given the reference strategy u^1, equal

$$\hat{u}^1(t) = u^2(t) + \frac{1}{k_{u^2|u^1}(t)}\left(u^1(t) - u^2(t)\right) = \frac{1}{2}u^1(t) + \frac{1}{2}u^2(t),$$

$$\hat{u}^2(t) = u^1(t) + \frac{1}{k_{u^1|u^2}(t)}\left(u^2(t) - u^1(t)\right) = \frac{1}{2}u^1(t) + \frac{1}{2}u^2(t),$$

respectively. In other words, an optimal Kelly trader (maximizing the rate of logarithmic return) will always hold half his wealth in each strategy. By contrast, for a Markowitz trader who bases his decisions on the rate of return process the situation is different. In particular, as shown in Theorem 5.12, we have

$$b_{u^1|u^1+k_1(\hat{u}^2-u^1)}(t) = k_1(t)s^2_{u^1|u^2}(t), \quad b_{u^2|u^2+k_2(\hat{u}^1-u^2)}(t) = k_2(t)s^2_{u^2|u^1}(t).$$

For a Markowitz trader, with reference strategy u^1, we see that when leveraging (shortening) u^2 by applying a positive (negative) Kelly multiplier k_1 the rate of return is positive (negative). However, for a Markowitz trader, with reference strategy u^2, leveraging (shortening) u^1, through a positive (negative) Kelly multiplier k_2, implies a positive (negative) rate of return. But this is inconsistent since leveraging, say, u^1 is the same as shortening u^2 and vice versa. Hence, the only self-financing strategies $w = (1-\lambda)u^1 + \lambda u^2$ where both Markowitz traders agree on making profits are those for which $0 < \lambda < 1$. This follows from the identifications $k_1 = 2\lambda$ and $k_2 = 2(1-\lambda)$. However, similar to Fig. 5 we see that for a fixed value of $\lambda \in (0, 1)$ the corresponding Kelly traders cannot be simultaneously efficient unless $\lambda = 1/2$.

5.3 A Note on Symmetries

We saw in Example 5.17 how the Markowitz approach of forming decisions based on the rate of return process $b_{u|w}$ can lead to internal contradictions. However, when decisions are based on the rate of logarithmic return process $\mu_{u|w}$, as in the Kelly approach, these contradictions disappear. In this section we take a closer look at these two processes and provide arguments for why one approach is, in many ways, better than the other.

Our approach make use of the unique decomposition of a bi-variate function into an anti-symmetric and a symmetric part. That is, suppose we are given a function $f : U \times U \to \mathbb{R}$, for some set U. If we define

$$f^{asym}(x,y) = \frac{1}{2}\left(f(x,y) - f(y,x)\right), \quad f^{sym}(x,y) = \frac{1}{2}\left(f(x,y) + f(y,x)\right), \tag{40}$$

it is easily seen that f^{asym} is anti-symmetric while f^{sym} is symmetric. Moreover, the decomposition $f = f^{asym} + f^{sym}$ is unique. We now apply this notation to the processes $b_{u|w}$, $\mu_{u|w}$ and $\sigma_{u|w}$ as shown below

$$\begin{aligned} b^{asym}_{u|w}(t) &= \mu_{u|w}(t), & b^{sym}_{u|w}(t) &= \frac{1}{2}\sigma^2_{u|w}(t), \\ \mu^{asym}_{u|w}(t) &= \mu_{u|w}(t), & \mu^{sym}_{u|w}(t) &= 0, \\ \sigma^{asym}_{u|w}(t) &= 0, & \sigma^{sym}_{u|w}(t) &= \sigma_{u|w}(t). \end{aligned}$$

What this schematic picture shows is that $\mu_{u|w}$ and $\sigma_{u|w}$ are the fundamental variables while $b_{u|w}$ is a derived quantity. In other words, while the volatility process cannot be explained by the rate of logarithmic return process, it would be possible to replace $\mu_{u|w}$ and $\sigma_{u|w}$ with expressions involving $b_{u|w}$ and $b_{w|u}$ throughout this paper. However, such an approach would, of course, be utterly confusing and misleading. In fact, the entire instantaneous covariance process V_u can be expressed in terms of either the volatility process or the rate of return process as described below.

Lemma 5.18. *Given an arbitrary $\mathbb{F}$-predictable reference strategy u and let w^1, w^2 be any $\mathbb{F}$-predictable trading strategies. Then*

$$V_{u|w^1,w^2}(t) = \frac{1}{2}\left(\sigma^2_{u|w^1}(t) + \sigma^2_{u|w^2}(t) - \sigma^2_{w^1|w^2}(t)\right) = b_{u|w^2}(t) + b_{w^1|u}(t) - b_{w^1|w^2}(t).$$

Proof. The first identity follows directly from Corollary 5.16, while the last identity follows from Eq. (39), after the change of variables $(u^1, u^2, w) \to (w^1, u, w^2)$, and the property $\sigma = \sigma^{sym}$. □

We conclude that modeling of risky assets should be done in terms of the rate of logarithmic return μ and not in terms of the rate of return b. This statement is further supported by Proposition 2.2 where we showed that the long term performance of any trading strategy is ultimately determined by the time average of the instantaneous rate of logarithmic return.

The decomposition introduced above can also be applied to the relative drawdown risk process and to the relative leverage risk process. Different to the volatility measure we see that relative drawdown risk is a pure anti-symmetric measure with

$$R^{asym}_{u|w}(t) = R_{u|w}(t), \quad R^{sym}_{u|w}(t) = 0. \tag{41}$$

By contrast, the relative leverage risk process has both anti-symmetrical and symmetrical components

$$k^{asym}_{u|w}(t) = \frac{2R_{u|w}(t)}{1 - R^2_{u|w}(t)}, \quad k^{sym}_{u|w}(t) = -\frac{2R^2_{u|w}(t)}{1 - R^2_{u|w}(t)}. \tag{42}$$

Alternatively, we can express the anti-symmetrical and symmetrical components of the relative leverage risk process using the decomposition for the relative leverage aversion process $1/k_{u|w}$. If we ignore the scaling factor (and the degenerate case where $u = w$) we obtain the simple expressions

$$\frac{1}{k_{u|w}(t)} - \frac{1}{k_{w|u}(t)} = \frac{1}{R_{u|w}(t)}, \quad \frac{1}{k_{u|w}(t)} + \frac{1}{k_{w|u}(t)} = 1. \tag{43}$$

Finally, we stress that the correct way to view the instantaneous generalized Sharpe ratio is through the formula

$$s_{u|w}(t) = \frac{\mu_{u|w}(t)}{\sigma_{u|w}(t)} + \frac{1}{2}\sigma_{u|w}(t) = s^{asym}_{u|w}(t) + s^{sym}_{u|w}(t). \tag{44}$$

Hence, for $\mu_{u|w}$ fixed, the instantaneous generalized Sharpe ratio explodes (and thereby generating arbitrage opportunities) should the volatility $\sigma_{u|w}$ tend to either zero or infinity. Bermin and Holm [3] show that, when $u = \mathbf{0}$, Kelly traders automatically force the portfolio volatility process and in particular the correlation process associated with V_0 towards a trading equilibrium. However, instead of repeating their rather lengthy arguments, for an arbitrary reference strategy u, we present below a different approach supporting their claims.

5.4 Relative Drawdown Risk and Correlation

In this section we show how the instantaneous correlation between two arbitrary portfolios can conveniently be identified when the relative drawdown risk is consistently traded. We use the notation

$$V_{u|w^1,w^2}(t) = \sigma_{u|w^1}(t)\rho_{u|w^1,w^2}(t)\sigma_{u|w^2}(t), \tag{45}$$

when referring to the instantaneous correlation process.

Proposition 5.19. *Given an arbitrary $\mathbb{F}$-predictable reference strategy u and let v^1, v^2 be any $\mathbb{F}$-predictable trading strategies such that $R_{u|v^1} = R_{u|v^2}$. Then*

$$\rho_{u|v^1,v^2}(t) = \begin{cases}\sigma_{u|v^1}(t)/\sigma_{u|v^2}(t) \\ \sigma_{u|v^2}(t)/\sigma_{u|v^1}(t)\end{cases} \Leftrightarrow R_{u|v^1}(t) = R_{u|v^2}(t) = \begin{cases}R_{v^1|v^2}(t) \\ R_{v^2|v^1}(t)\end{cases}.$$

Furthermore, if $R_{u|v^1} = R_{u|v^2} \neq -1$ we have the alternative characterization

$$\sigma_{u|v^1}(t)/\sigma_{u|v^2}(t) = s_{u|v^1}(t)/s_{u|v^2}(t).$$

Proof. Since v^1, v^2 have the same relative drawdown risk it follows from Theorem 5.12 that there exists a common Kelly multiplier such that

$$v^1(t) = u(t) + k(t)\left(\hat{v}^1(t) - u(t)\right), \quad v^2(t) = u(t) + k(t)\left(\hat{v}^2(t) - u(t)\right).$$

Hence, we can view these trading strategies as generalized Kelly strategies. By combining Proposition 5.14 and Lemma 5.18 we find

$$\begin{aligned}\rho_{u|v^1,v^2}(t) &= \frac{1}{2}\left(\frac{\sigma_{u|v^1}(t)}{\sigma_{u|v^2}(t)} + \frac{\sigma_{u|v^2}(t)}{\sigma_{u|v^1}(t)} - \frac{\sigma^2_{v^1|v^2}(t)}{\sigma_{u|v^1}(t)\sigma_{u|v^2}(t)}\right),\\ &= \frac{1}{2}\left(\frac{\sigma_{u|v^1}(t)}{\sigma_{u|v^2}(t)}\left(1 + \frac{R_{v^1|v^2}(t)}{R_{u|v^1}(t)}\right) + \frac{\sigma_{u|v^2}(t)}{\sigma_{u|v^1}(t)}\left(1 - \frac{R_{v^1|v^2}(t)}{R_{u|v^2}(t)}\right)\right).\end{aligned}$$

The first result then follows from studying the solutions under the assumption that $R_{u|v^1} = R_{u|v^2}$, while making use of the identity $R_{v^2,v^1} = -R_{v^1,v^2}$. The proof concludes from the observation that

$$s_{u|w}(t) = \frac{1}{2}\left(1 + \frac{1}{R_{u|w}}\right)\sigma_{u|w}(t),$$

for any trading strategy w. □

It is interesting to note that we recover the expression for correlation equilibrium in [3] by simply requiring the relative drawdown risk to satisfy a triangle consistency criteria. Our approach further highlights the fact that when the relative drawdown risk between two arbitrary portfolios satisfies the triangle consistency criteria the correlation cannot be negative. Note further that the two branches in the result above are mutually exclusive since the magnitude of the correlation is bounded by one.

Example 5.20. In order to illustrate the notion of trading equilibrium first presented in [3] we consider a reduced market consisting of two risky assets and the market numéraire asset. Straightforward calculations, using Theorem 3.2, then verify that the growth optimal Kelly strategy equals

$$\arg\max_w \mu_{0|w}(t) = \frac{1}{1 - \rho^2_{0|e^1,e^2}(t)}\left(\frac{s_{0|e^1}(t) - \rho_{0|e^1,e^2}(t)s_{0|e^2}(t)}{\sigma_{0|e^1}(t)}, \frac{s_{0|e^2}(t) - \rho_{0|e^1,e^2}(t)s_{0|e^1}(t)}{\sigma_{0|e^2}(t)}\right)'.$$

We further assume that the two assets are traded consistently with respect to the market numéraire; that is we assume that there exists a process $R \neq -1$ such that $R_{0|e^1} = R_{0|e^2} = R$. This means that the individual relative leverage processes satisfy $k_{0|e^1} = k_{0|e^2} = 2R/(1+R)$. We let k denote the common relative leverage process and note, from Theorem 4.1, that $e^1 = k\hat{e}^1$ and $e^2 = k\hat{e}^2$. Without loss of generality we now assume that $\sigma_{0|e^2} > \sigma_{0|e^1}$ and consider the case where the instantaneous correlation process $\rho_{0|e^1,e^2} = \sigma_{0|e^1}/\sigma_{0|e^2}$. It then follows from Proposition 5.19 that $\rho_{0|e^1,e^2} = s_{0|e^1}/s_{0|e^2}$ which implies that the growth optimal Kelly strategy $w^* = \hat{e}^2$. Hence, an investor who applies the same Kelly multiplier to the growth optimal Kelly strategy as to the single name strategies will set $w = kw^* = e^2$.

We now compare this allocation with the trading strategies set by an investor using the reference strategy $u = e^1$. If such an investor can trade in both the risky asset and the market numéraire the corresponding Kelly strategy takes the

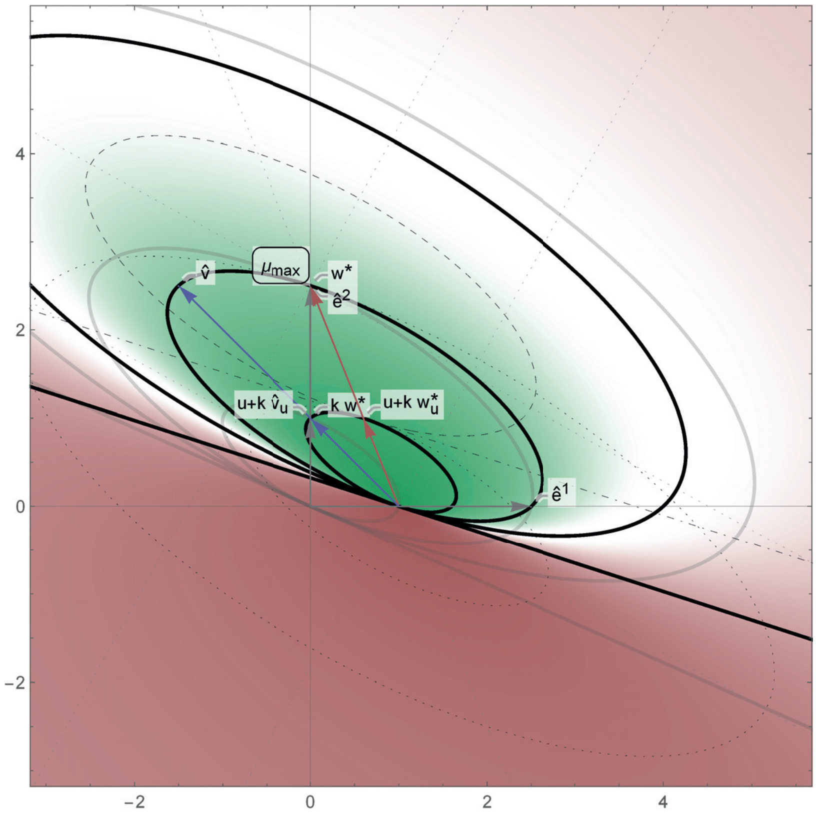

Figure 6: This picture highlights the trading equilibrium corresponding to the instantaneous correlation process $\rho_{0|e^1,e^2} = \sigma_{0|e^1}/\sigma_{0|e^2}$, where $\sigma_{0|e^2} > \sigma_{0|e^1}$. Under the additional assumption that $k_{0|e^1} = k_{0|e^2} = k$, the growth optimal Kelly strategy $w^* = \hat{e}^2$. An investor, with reference strategies $u = \mathbf{0}$, who applies the Kelly multiplier k to the growth optional Kelly strategy will consequently set $w = kw^* = e^2$. This trading strategy is identical to the one obtained by an investor, with reference strategies $u = e^1$, who applies the Kelly multiplier k to a generalized Kelly strategy $\hat{v}$ which only trades in the two risky assets (blue arrow). We also highlight the optimal trading strategy, with respect to the reference strategies $u = e^1$, for an investor who can trade in both the risky assets and in the market numéraire (red arrow).

form $w = e^1 + k(\hat{e}^2 - e^1)$. This trading strategy is represented by the red arrow in Fig. 6. Next, we consider the situation where the investor can only trade in the two risky assets. We set $w = e^1 + k(\hat{v} - e^1)$ for some Kelly multiplier k and some trading strategy $\hat{v}$ such that $w_1 + w_2 = 1$. Straightforward calculations then show that any trading strategy v of the form $v = xe^1 + (1 - x)e^2$, for some real-valued $\mathbb{F}$-predictable process x, meets the requirement. By the use of Lemma 5.3 and Definition 5.4 we now compute $k_{e^1|xe^1+(1-x)e^2} = (1 - x)k_{e^1|e^2}$, which together with Theorem 5.12 yields

$$w(t) = e^1(t) + \frac{k(t)}{k_{e^1|e^2}(t)}(e^2(t) - e^1(t)).$$

This trading strategy is represented by the blue arrow in Fig. 6. Finally, it follows from Proposition 5.19 that $k_{e^1|e^2} = k$ when the instantaneous correlation $\rho_{0|e^1,e^2} = \sigma_{0|e^1}/\sigma_{0|e^2}$, $\sigma_{0|e^2} > \sigma_{0|e^1}$. Hence, in this case, we obtain the same allocation $w = e^2$ as we did for a Kelly trader with reference strategy $u = \mathbf{0}$.

6 How to Beat an Index

We end this paper with the explicit application of how to beat an index. More precisely, we investigate how to maximize the logarithmic return in excess of that for the index, given an arbitrary leverage target. In fact this is almost precisely what Theorem 5.5 is about and the steps to take are:

- Identify the reference strategy u and its opportunity set.
- Potentially add risky assets to the opportunity set of the reference strategy.
- Compute the growth optimal Kelly strategy w^* for the extended opportunity set.
- Choose a risk level $R > 0$ and set the Kelly multiplier $k = 2R/(1+R)$ accordingly.
- Form the Kelly strategy $w = u + k(w^* - u)$.

The Kelly strategy described above is guaranteed to have maximal generalized Sharpe ratio. However, this does not necessarily mean that for a fixed relative drawdown risk level the Kelly strategy is the one with maximal rate of logarithmic return. We provide the missing detail below.

Proposition 6.1. *Given an arbitrary $\mathbb{F}$-predictable reference strategy u and let R denote an $\mathbb{F}$-predictable risk target. Then*

$$\underset{w \ s.t.\ R_{u|w}=R>0}{\arg\max} \mu_{u|w}(t) = u(t) + \frac{2R(t)}{1+R(t)} \left(w^*(t) - u(t)\right),$$
$$\underset{w \ s.t.\ R_{u|w}=R<0}{\arg\min} \mu_{u|w}(t) = u(t) + \frac{2R(t)}{1+R(t)} \left(w^*(t) - u(t)\right).$$

Proof. Let v be an arbitrary trading strategy and define

$$w(t) = u(t) + k(t)\left(\hat{v}(t) - u(t)\right), \quad k(t) = \frac{2R(t)}{1+R(t)}.$$

Then, as shown in Theorem 5.12, w is a generalized Kelly strategy with $R_{u|w} = 1/\mathcal{A}(k) = R$. Note further that any trading strategy w for which $R_{u|w} = R$ takes the form above for some trading strategy $\hat{v}$ such that $k_{u|\hat{v}} = 1$. By combining Theorem 5.5 and Theorem 5.12 we therefore get

$$|\mu_{u|w}(t)| = \frac{1}{2}|k(t)(2-k(t))|s^2_{u|\hat{v}}(t) \leq \frac{1}{2}|k(t)(2-k(t))|s^2_{u|w^*}(t) = |\mu_{u|w^*}(t)|.$$

We finally observe that $k(2-k) > 0 \Leftrightarrow k \in (0,2) \Leftrightarrow R > 0$. This concludes the proof. ☐

The above result shows that one should always use a Kelly strategy when trying to beat an arbitrary reference strategy. Not only will such a strategy yield the highest rate of logarithmic return for every fixed risk level $R > 0$, it also yields the maximal generalized Sharpe ratio.

In Fig. 7 we consider the case where an investor can trade in S&P500, gold and the US-bank account. The reference strategies equal either the US-bank account, $u = \mathbf{0}$, or the total return index of S&P500, $u = e^1$. In order to derive the local characteristics of the corresponding Kelly strategies we first notice, by the use of Definition 5.1 and Lemma 5.3, that we can relate the generalized Sharpe ratio (with respect to u) of the optimal Kelly strategy to the volatility (with respect to $u = \mathbf{0}$) of a portfolio long the optimal Kelly strategy and short the reference strategy, according to

$$s^2_{u|w^*}(t) = \sigma^2_{0|w^*-u}(t) = \langle w^* - u, w^* - u\rangle_{V_0}(t). \tag{46}$$

Straightforward calculations, using the estimated parameters in Fig. 1, then yield $s_{0|w^*} = 0.665$ and $s_{e^1|w^*} = 0.543$. Note that, in each case, these are the maximal generalized Sharpe ratios for the given reference strategies. For comparison we also state that the generalized Sharpe ratio of the total return index of S&P500, with respect to the US-bank account, equals $s_{0|e^1} = 0.476$. Hence, by adding gold to the opportunity the generalized Sharpe ratio can be improved.

w	$k_{0\|w}$	$R_{0\|w}$	$s_{0\|w}$	$\mu_{0\|w}$	$\sigma_{0\|w}$	$k_{e^1\|w}$	$R_{e^1\|w}$	$s_{e^1\|w}$	$\mu_{e^1\|w}$	$\sigma_{e^1\|w}$
e^1	0.412	0.259	0.476	0.074	0.196	0	0	0	0	0
$k_{0\|e^1}w^*$	0.412	0.259	0.665	**0.145**	0.274	0.412	0.259	0.465	0.071	0.191
$e^1 + k_{0\|e^1}w^*_{e^1}$	0.563	0.392	0.649	0.170	0.365	0.412	0.259	0.543	**0.096**	0.223
$e^1 + kw^*_{e^1}$	0.412	0.259	0.570	0.106	0.235	0.116	0.061	0.543	**0.032**	0.063

Table 1: Portfolio characteristics, with Kelly multiplier $k = 0.116$, for the opportunity set: S&P500, gold and US-bank account. We use the estimated parameters in Fig. 1 such that for the fixed risk level $R_{u|w} = 0.259$, $u \in \{\mathbf{0}, e^1\}$, we have $\max_w \mu_{0|w} = 0.145$ and $\max_w \mu_{e^1|w} = 0.096$. Note further that $\max_w \mu_{e^1|w} = 0.032$ for the fixed risk level $R_{e^1|w} = 0.061$.

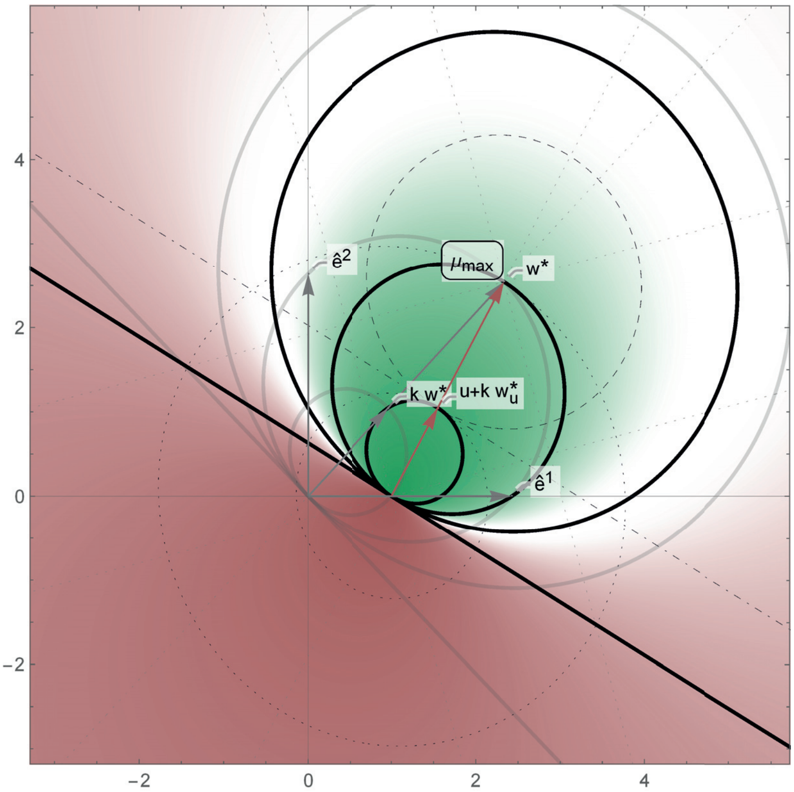

Figure 7: This plot shows the growth optimal Kelly strategy when the reference strategy equals the total return index of S&P500, $u = e^1$, based on the historical estimates reported in Fig. 1. We highlight the level curves of leverage risk for $k_{u|w} \in \{k_{0|e^1}, 1, 2, \pm\infty\}$.

However, in order to calculate the exact improvement we must fix the reference strategy and the Kelly multiplier. In Table 1 we consider various trading strategies, including the strategy $w = e^1 + k(w^* - e^1)$, where the Kelly multiplier k is chosen such as to match the leverage risk of S&P500 relative to the US-bank account

$$k_{0|e^1}(t) = k_{0|e^1+k(w^*-e^1)}(t). \tag{47}$$

In this case, it is easily seen that k solves a quadratic equation with $k = 0$ being one solution. Straightforward algebraic manipulations further verify that the second solution equals

$$k(t) = 1 - (1 - k_{0|e^1})\frac{s^2_{0|w^*}(t)}{s^2_{e^1|w^*}(t)}. \tag{48}$$

Table 1 verifies that one should always use a Kelly strategy, as in Proposition 6.1, when trying to beat an index. There is, however, great flexibility in how to set the targeted risk level as demonstrated above. We also observe that $k_{e^1|w} = k_{0|w} = k_{0|e^1}$ for the trading strategy $w = k_{0|e^1}w^*$, as proved in Proposition 5.11.

7 Conclusions

In this paper we build a comprehensive framework that allows an investor to analyze leverage risk for arbitrary trading strategies and for arbitrary reference assets (or portfolios). Key to our framework is the notion of relative drawdown risk where relative means relative to an arbitrary chosen reference asset. With this viewpoint there is no such thing as an objective risk-free asset. For instance, an investor who holds dollar as his reference asset considers euro as a risky investment and interchangeably an investor who holds euro as his reference asset considers dollar as a risky investment. No view is better then the other; it is just a matter of what is the reference asset and, thereby, what asset is considered risk-free. So the immediate question is: what if one were to choose a reference portfolio of currencies with, say, fifty percent in dollar and fifty percent in euro. How would one calculate drawdown risk in such a case? This is one of the main questions we address and derive the solution for in this article. We also characterize the set of all trading strategies facing the same relative drawdown risk and additionally we show how to translate an investor's viewpoint from one choice of reference asset (or portfolio) to another.

The other main question we attempt to answer is: how to beat an index. In order to answer this question we first prove that the growth optimal Kelly strategy is the only trading strategy for which the relative drawdown risk is not dependent on the choice of the reference asset. This implies that an investor trying to beat an index should always invest a fraction of his wealth in the growth optimal Kelly strategy and the remaining wealth in the index. The particular fraction chosen to invest in the growth optimal Kelly strategy depends on the drawdown risk relative to the index that the investor targets. The fact that such a simple linear trading rule is locally efficient is quite remarkable.

We also show that, for a given reference asset, the correlation between two arbitrary portfolios with identical relative drawdown risk equals the ratio of their generalized Sharpe ratios if and only if the relative drawdown risk is traded consistently. This observation supports a claim in [3] where such a result is derived, albeit through different methods, as a trading equilibrium. More surprisingly, we find that leverage applied to the growth optimal Kelly strategy affects the relative drawdown risk in much the same way as the speed of light affects velocities in Einstein's theory of special relativity.

A Appendix

In this section we provide the proof of Proposition 2.5. The results derived are standard applications of Feller's test for explosions and for ease of reading we use the terminology of [10] as much as possible.

Proof. We start by noticing that a portfolio for which the the relative drawdown aversion is held constant evolves according to

$$d\log X_{0|w}(t) = \frac{1}{2}A\sigma^2_{0|w}(t)dt + \sigma_{0|w}(t)dW(t), \quad X_{0|w}(0) = x_0,$$

for some Brownian motion W. This follows from our initial assumption that the processes governing the risky assets have continuous sample paths. Furthermore, the stochastic process $\bar{W}$ defined through the relationship

$$\bar{W}\left(t\bar{\sigma}^2_{0|w}(t)\right) = \int_0^t \sigma_{0|w}(s)dW(s), \quad \lim_{t\to\infty} t\bar{\sigma}^2_{0|w}(t) = \infty \quad a.s.$$

is again a Brownian motion according to Lévy's theorem [10]. We now introduce the process

$$dY(t) = \frac{1}{2}Adt + d\bar{W}(t), \quad Y(0) = 0,$$

such that

$$\log\frac{X_{0|w}(t)}{x_0} = Y(h(t)), \quad h(t) = t\bar{\sigma}^2_{0|w}(t) = \int_0^t \sigma^2_{0|w}(s)ds.$$

This shows that with $\eta_{a,b} = \inf\{t \geq 0 : Y(t) \notin (\log a, \log b)\}$ we have $\eta_{a,b} = h(\tau_{0|a,b})$. Hence, $\eta_{a,b}$ is a.s. finite (infinite) if and only if $\tau_{0|a,b}$ is a.s. finite (infinite). Following [10] we define the scale function p and the corresponding function v associated with the process Y, using $Y(0) = 0$ as the reference point for convenience, according to

$$p(y) = \frac{1}{A}(1 - \exp(-yA)), \quad v(y) = \int_0^y p'(x)\int_0^x \frac{2dz}{p'(z)}dx. \tag{49}$$

For $a > 0$ and $b < \infty$ it follows that $p(\log a) > -\infty$ and $p(\log b) < \infty$. The probabilities of Y hitting either of the barriers $(\log a, \log b)$ can then be found in [10] (Proposition 5.22) and translate to the original problem according to

$$\mathbb{P}(X_{0|w}(\tau_{0|a,b}) = cx_0) = \mathbb{P}(Y\left(\eta_{0|a,b}\right) = \log c), \quad c \in \{a, b\}.$$

We further calculate

$$v(y) = \frac{2}{A}(y - p(y)),$$

from which we see that $v(\log a), v(\log b) < \infty$. From [10] (Propositions 5.32) we then conclude that $\mathbb{E}[\eta_{a,b}] < \infty$, which together with Doob's optional sampling theorem yields

$$\mathbb{E}[Y(\eta_{a,b})] = \frac{1}{2}A\mathbb{E}[\eta_{a,b}].$$

The expression for $\mathbb{E}[\eta_{a,b}]$ can now be derived from the alternative representation

$$\mathbb{E}[Y(\eta_{a,b})] = \mathbb{P}(Y(\eta_{a,b}) = \log a) \log a + \mathbb{P}(Y(\eta_{a,b}) = \log b) \log b.$$

Note further that $\mathbb{E}[\eta_{a,b}] < \infty$ implies that $\eta_{a,b}$ is a.s. finite, which in turn implies that $\tau_{0|a,b}$ is a.s. finite.

In the limit as $a \to 0$ and $b \to \infty$ we obtain $v(\pm\infty) = \infty$. Feller's test for explosions, see [10] (Theorem 5.29), then states that $\eta_{0,\infty}$ (and hence $\tau_{0|0,\infty}$) is infinite with probability one. In order to characterize explosions from the intervals $(0, b)$, $b < \infty$, and (a, ∞), $a > 0$, we compute $p(-\infty) = -\infty$, $A \geq 0$, and $p(\infty) = \infty$, $A \leq 0$. From [10] (Proposition 5.32) we then conclude that $\eta_{0,b} < \infty$ a.s. for $A \geq 0$ and that $\eta_{a,\infty} < \infty$ a.s. for $A \leq 0$. $\square$

References

[1] Artzner, P., Delbaen, F., Eber, J., Heath, D.: Coherent Measures of Risk. Mathematical Finance (1999) doi: 10.1111/1467-9965.00068

[2] Asness, C.S, Frazzini, A., Pedersen, L.H.: Leverage Aversion and Risk Parity. Financial Analysts Journal (2018) doi: 10.2469/faj.v68.n1.1

[3] Bermin, H.-P., Holm, M.: Kelly Trading and Market Equilibrium. Working Paper, Knut Wicksell Centre for Financial Studies, Lund University (2019)

[4] Clarke, R., De Silva, H., Thorley, S.: Portfolio constraints and the fundamental law of active management. Financial Analysts Journal **58**, 48-–66 (2002)

[5] Davis, M., Lleo, S., 2013. Fractional Kelly Strategies in Continuous Time: Recent Developments, In: MacLean, L. C., Ziemba, W. T. (Eds.), Handbook of the Fundamentals of Financial Decision Making. World Scientific, Chapter 37, 753–787. https://doi.org/10.1142/9789814417358-0037.

[6] Föllmer, H., Schied, A.: Convex measures of risk and trading constraints. Finance and Stochastics (2002) doi: 10.1007/s007800200072

[7] Jacobs, B., Levy, K.: Leverage aversion and portfolio optimality. Financial Analysts Journal **68**, 89-–94 (2012)

[8] Jacobs, B., Levy, K.: Leverage aversion, efficient frontiers, and the efficient region. The Journal of Portfolio Management (2013) doi: 10.3905/jpm.2013.39.3.054

[9] Jacobs, B., Levy, K., Markowitz, H.: Portfolio optimization with factors, scenarios, and realistic short positions. Operations Research **53**, 586-–599 (2005)

[10] Karatzas, I., Shreve, S.E.: Brownian Motion and Stochastic Calculus. Springer-Verlag, New York Heidelberg Berlin (1988)

[11] Kelly, J.L.: A new interpretation of information rate. The Bell System Technical Journal (1956) doi: 10.1002/j.1538- 7305.1956.tb03809.x

[12] Latané, H.: Criteria for choice among risky assets. J. Polit. Econ. (1959) doi: 10.1086/258157

[13] Liptser, R., Shiryayev, A.N.: Theory of Martingales. Kluwer Academic Publishers, Dordrecht (1989)

[14] MacLean, L. C., Ziemba, W. T., Blazenko, G., 1992. Growth versus security in dynamic investment analysis. Management Science. 38(11), 1562–1586. https://doi.org/10.1287/mnsc.38.11.1562.

[15] Markowitz, H.: Portfolio Selection. Journal of Finance (1952) doi: 10.1111/j.1540-6261.1952.tb01525.x

[16] Markowitz, H.: The optimization of a quadratic function subject to linear constraints. Naval Research Logistics Quarterly (1956) doi: 10.1002/nav.3800030110

[17] Markowitz, H.: How to represent mark-to-market possibilities with the general portfolio selection model. The Journal of Portfolio Management (2013) doi: 10.3905/jpm.2013.39.4.001

[18] Merton, R., 1973. An Intertemporal Capital Asset Pricing Model. Econometrica. 41(5), 867–887. https://doi.org/10.2307/1913811

[19] Nielsen, L. T., Vassalou, M.: Sharpe Ratios and Alphas in Continuous Time. Journal of Financial and Quantitative Analysis (2004) doi: 10.1017/S0022109000003902

[20] Platen, E.: A Benchmark Approach to Finance. Mathematical Finance (2006) doi.org/10.1111/j.1467-9965.2006.00265.x

[21] Sharpe, W. F.: Mutual fund performance. Journal of Business **39**, 119–138 (1966)

[22] Sharpe, W. F.: The Sharpe Ratio. The Journal of Portfolio Management (1994) doi: 10.3905/jpm.1994.409501

[23] Sorensen, E.H., Hua, R., Qian, E.: Aspects of constrained long-short equity portfolios. The Journal of Portfolio Management (2007) doi: 10.3905/jpm.2007.674790

[24] Thorp, E. O.: Portfolio choice and the Kelly criterion. Proceedings of the Business and Economics Section of the American Statistical Association, 215-–224 (1971)

Leverage and risk relativity: how to beat an index

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In this paper we show that risk associated with leverage is fundamentally relative to an arbitrary relative to the chosen reference asset. We further prove that the growth optimal Kelly portfolio is the only portfolio for which the relative drawdown risk is not dependent on the choice of the reference asset. Additionally, we show how to translate an investor's viewpoint from one choice of reference asset to another and establish conditions for when two investors can be said to face identical leverage risk. We also prove that, for a given reference asset, the correlation between two arbitrary portfolios with identical leverage risk equals the ratio of their Sharpe ratios if and only if the leverage risk is consistently traded. More surprisingly, we observe that leverage applied to the growth optimal Kelly strategy affects the drawdown risk in much the same way as the speed of light affects velocities in Einstein's theory of special relativity. Finally, we provide details on how to trade in order to beat an arbitrary index for a given leverage risk target.

Keywords: Leverage, Drawdown risk, Generalized Kelly strategy, Numéraire invariance, Risk relativity

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