Kelly Trading and Market Equilibrium

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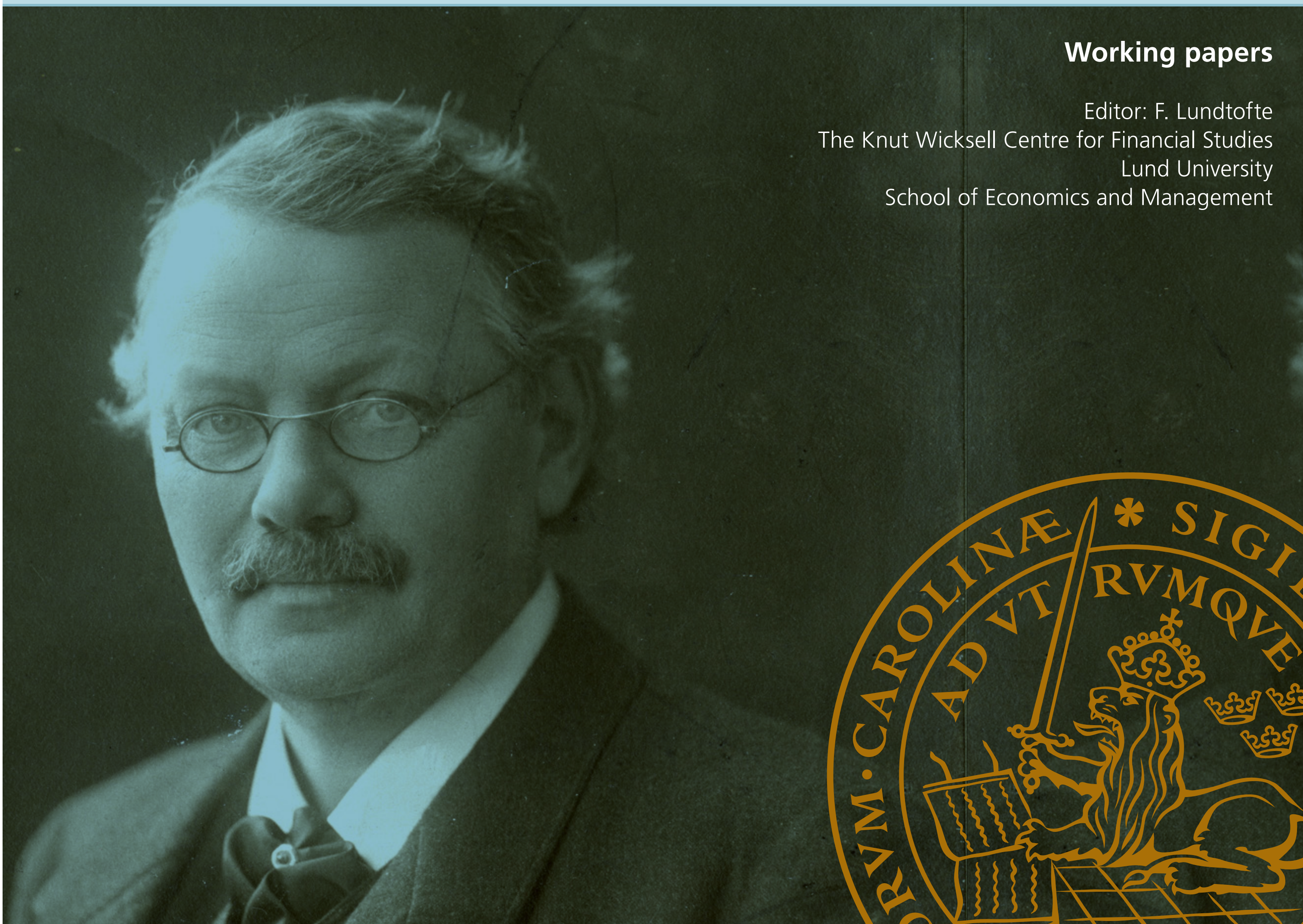

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KELLY TRADING AND MARKET EQUILIBRIUM

A PREPRINT

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ABSTRACT

We show that the Kelly framework is the natural multi-period extension of the one-period mean-variance model of Markowitz. Any allocation on the instantaneous Kelly efficient frontier can be reached by trading in the bank account and a particular mutual fund consisting of risky assets only. However, different to the mean-variance model there is an upper bound on the instantaneous Kelly efficient frontier. Any allocation surpassing this bound, called the optimal Kelly strategy, is known to be inefficient. We use the optimal Kelly strategy to deduce an expression for the instantaneous covariance matrix that results in market equilibrium. We show that this equilibrium is stable in the sense that any optimal Kelly trader, and most Kelly traders, will force the market covariance matrix towards equilibrium. We further investigate the role of a central bank setting the short-term interest rate. We show that an equilibrium rate exists but we argue that there is no market mechanism that will force the interest rate to this level. This motivates the short-term interest rate to be actively managed by the central bank.

Keywords Portfolio theory · Kelly criterion · Market equilibrium · Central bank rate

1 Introduction

The problem of portfolio selection largely falls into three main categories. The first of these is the generic expected utility approach of von Neumann and Morgenstern [18]. The second is the celebrated mean-variance approach developed by Markowitz [10], Tobin [16] and Merton [11]. The third approach of Kelly [8], Latané [9] and Breiman [2], [3] is the less known optimal growth theory, built upon results from betting and gambling. While the mean-variance approach is a one-period model the optimal growth theory relies on a genuine multi-period framework. What people engaged in betting have known for a long time is that capital accumulation stops the day bankruptcy occurs. Instead it is better to make frequent small gains that can be re-invested over time and let the compounding effect grow the capital. One way to describe such a strategy is to say that more emphasis should be put on maximizing the geometric mean of the individual bets rather than maximizing their arithmetic mean. This statement is often credited to Williams [19] and can be seen as the cornerstone for the work pioneered independently by Kelly and Latané. Many of the finer mathematical details regarding maximizing the geometric mean were subsequently worked out by Breiman. The majority of those result relates to long-term properties such as: the growth-optimal portfolio is the portfolio that minimizes the expected time to reach an arbitrary high monetary target, see [6] for a concise overview of optimal growth theory and its various applications. The term Kelly criterion, as a reference to the growth-optimal portfolio, is due to Thorp [15] who, among other things, placed the theory in a continuous time framework. In order to emphasize on the compounding effect to capital growth formation we have chosen to work with compounded rate of returns rather than geometric means. This allows us to study both continuously and discretely compounding rate of returns, a choice we feel is more inline with the current finance terminology.

While the theory of portfolio selection is of great interest for investors it also forms the basis for studying aggregate allocations and, in particular, market equilibrium. A standard approach is to assume that each investor chooses his

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portfolio allocation according to the mean-variance approach with an opportunity set consisting of risky assets and a bond. The optimal allocations can then be described by a straight line in the space spanned by the mean and the standard deviation of the return associated with the portfolio. This implies that any optimal combination of mean and standard deviation, known as the efficient frontier, yields the same Sharpe ratio. Furthermore, each allocation on the efficient frontier can be reached by trading in the bond and a specific mutual fund consisting of risky assets only. Such a separation further implies that when aggregating each investor's portfolio allocation we obtain a combination of mean and standard deviation that again has the same Sharpe ratio and thus can be represented as the solution to a mean-variance optimization problem, albeit for a representative agent. Hence, most results about aggregate allocations can be reduced to a single person optimal allocation problem. While it is standard to assume that such a representative individual is a mean-variance investor most aggregate results can often be generalized to economies where the market participants form their allocation decisions based on the expected utility approach, see [12] for an overview. However, little effort has been made in studying aggregate behaviour in the case where the market participants are growth-optimizers. The reason being, we envision, that aggregate phenomena like equilibrium is a short-term, or more precisely instantaneous, concept.

In this paper we aim to bridge the gap between short-term and long-term properties. We show that the Kelly framework is the natural multi-period extension of the one-period mean-variance approach. The instantaneous Sharpe ratio of the Kelly strategy gives rise to an instantaneous efficient frontier. Any allocation on the efficient frontier can be obtained by trading in the bank account and a particular mutual fund consisting of risky assets only. However, different to the mean-variance approach there is an upper bound on the Kelly efficient frontier. Any allocation surpassing this bound, called the optimal Kelly strategy, is known to be inefficient. We use the optimal Kelly strategy to deduce an expression for the instantaneous covariance matrix that results in market equilibrium. We show that this equilibrium is stable in the sense that any optimal Kelly trader, and most Kelly traders, will force the market covariance matrix towards equilibrium. We further investigate the role of a central bank setting the short-term interest rate. We show that an equilibrium rate exists but we argue that there is no market mechanism that will force the interest rate to this level. This motivates the short-term interest rate to be actively managed by the central bank.

2 Modeling the Market

We consider a capital market consisting of a bank account B and a number of assets $P = (P_1, \ldots, P_N)'$. An asset related to a dividend paying stock is seen as a fund with the dividends re-invested. All assets are assumed to be adapted stochastic processes living on a filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$, where $\mathbb{F} = \{\mathcal{F}(t) : t \geq 0\}$ is a right-continuous increasing family of σ-algebras such that $\mathcal{F}(0)$ contains all the $\mathbb{P}$-null sets of $\mathcal{F}$. As usual we think of the filtration $\mathbb{F}$ as the carrier of information.

An investor can trade the assets and for the sake of simplicity we assume that there are no transaction fees associated with trading, that short-selling is allowed, that trading takes place continuously in time, and that the investor's trading activity does not impact the asset prices. We define a trading strategy as an $\mathbb{F}$-predictable process $q = (q_0, q_1, \ldots, q_N)'$, representing the number of shares held in each asset, such that the corresponding portfolio process takes the form

$$X(t) = q_0(t) B(t) + \sum_{n=1}^{N} q_n(t) P_n(t). \tag{1}$$

In order to analyze the performance of the portfolio process we impose the restriction that, when re-balancing the portfolio, money can neither be injected nor withdrawn. Such trading strategies are said to be self-financing and imply that the portfolio process evolves according to

$$X(t) = X(0) + \int_0^t q_0(s)\, dB(s) + \sum_{n=1}^{N} \int_0^t q_n(s)\, dP_n(s). \tag{2}$$

In many applications it is often more convenient to define the self-financing trading strategy as being proportional to the wealth X. For this reason we introduce $w = (w_1, \ldots, w_N)'$ according to

$$w_n(t) X(t) = q_n(t) P_n(t), \quad 1 \leq n \leq N, \tag{3}$$

such that, when expressed in differential form, the portfolio dynamics equals

$$\frac{dX(t)}{X(t)} = \left(1 - \sum_{n=1}^{N} w_n(t)\right) \frac{dB(t)}{B(t)} + \sum_{n=1}^{N} w_n(t) \frac{dP_n(t)}{P_n(t)}. \tag{4}$$

Moving forward we must introduce asset models to describe the evolution of the capital market. In order to strive for full generality we assume that all stochasticity is generated by an M-dimensional standard Brownian motion W, where $M \geq N$, and we identify the filtration $\mathbb{F}$ with the $\mathbb{P}$-augmentation of the natural filtration of W. We assume that the bank account is locally risk-free with

$$dB(t) = B(t)\, r(t)\, dt, \tag{5}$$

and thus fully determined by the $\mathbb{F}$-adapted interest rate process r. Regarding the risky assets we set

$$\frac{dP_n(t)}{P_n(t)} = b_n(t)\, dt + \Sigma_n'(t)\, dW(t), \quad 1 \leq n \leq N, \tag{6}$$

where the local rate of return vector $b = (b_1, \ldots, b_N)'$ and the volatility matrix $\Sigma = (\Sigma_1, \ldots, \Sigma_N)'$ are arbitrary $\mathbb{F}$-adapted processes. Note that each volatility vector process Σ_n, $1 \leq n \leq N$, takes values in $\mathbb{R}^M$. We also let $\sigma = (\|\Sigma_1\|, \ldots, \|\Sigma_N\|)'$ be the vector of real-valued asset volatilities and write $\sigma_{diag} = \operatorname{diag}(\sigma)$ for the associated diagonal matrix. The instantaneous asset-asset covariance matrix V can then be expressed in terms of the corresponding instantaneous correlation matrix ρ according to

$$V(t) = \Sigma(t)\, \Sigma'(t) = \sigma_{diag}(t)\, \rho(t)\, \sigma_{diag}(t). \tag{7}$$

For the purpose of this paper we always assume that V is an a.s. positive definite matrix. This implies that the inverse of V exists and that V^{-1} is a.s. positive definite. Hence, each component of σ is a.s. strictly positive, which allows us to introduce the instantaneous Sharpe ratios $s = (s_1, \ldots, s_N)'$, as defined in [13], according to

$$s(t) = \sigma_{diag}^{-1}(t)\, (b(t) - r(t)\, \mathbf{1}_N), \quad \mathbf{1}_N = (1, \ldots, 1)'. \tag{8}$$

Given the model parameters (b, Σ), alternatively (s, σ), and the interest rate r the evolution of the portfolio process then takes the form

$$\frac{dX(t)}{X(t)} = (r(t) + w'(t)\, \sigma_{diag}(t)\, s(t))\, dt + w'(t)\, \Sigma(t)\, dW(t), \tag{9}$$

such that the local rate of return b_X and the volatility Σ_X of the portfolio process can be expressed as

$$b_X(t) = r(t) + w'(t)\, \sigma_{diag}(t)\, s(t), \quad \Sigma_X(t) = \Sigma'(t)\, w(t). \tag{10}$$

If we further set

$$\sigma_X^2(t) = \|\Sigma_X(t)\|^2 = w'(t)\, V(t)\, w(t), \tag{11}$$

the instantaneous Sharpe ratio of the portfolio equals

$$s_X(t) = \frac{b_X(t) - r(t)}{\sigma_X(t)}, \tag{12}$$

with the convention that $s_X = 0$ if $b_X = r$ and $\sigma_X = 0$.

Having specified the evolution of the portfolio process we now turn to evaluating the trading performance. For any trader this equates to measuring the return of his portfolio. We argue that a trader is interested in the compounded return rather than the periodic return, since early financial gains can be re-invested and made to work for a longer period of time. Following market conventions we express the various return types as annualized rates such that, for instance, the continuously compounded rate of return, or as it is also called the logarithmic rate of return, is given by

$$R_X^{cc}(T; S) = \frac{1}{T - S} \log\left(\frac{X(T)}{X(S)}\right), \quad 0 \leq S \leq T, \tag{13}$$

while the periodic rate of return equals

$$R_X^{p}(T; S) = \frac{1}{T - S} \frac{X(T) - X(S)}{X(S)}, \quad 0 \leq S \leq T. \tag{14}$$

In order to keep the notation simple we often write $R_X^{cc}(T)$ instead of $R_X^{cc}(T; 0)$ when it is clear that we measure the rate of return from today's date and similar for the periodic rate of return. A further study of the logarithmic wealth requires the use of the Itô formula. Straightforward calculations yield

$$d \log(X(t)) = \left(b_X(t) - \frac{1}{2}\sigma_X^2(t)\right) dt + \Sigma_X'(t)\, dW(t). \tag{15}$$

We now let μ_X denote the drift of $\log(X)$ such that

$$\mu_X(t) = b_X(t) - \frac{1}{2}\sigma_X^2(t). \tag{16}$$

Similarly, we let $\mu_n = b_n - \frac{1}{2}\sigma_n^2$ denote the logarithmic drift of the risky asset P_n and set $\mu = (\mu_1, \ldots, \mu_N)'$. It is worth to stress the obvious fact that, given σ_X, the terms b_X and μ_X cannot be specified simultaneously. For sake of mathematical elegance it is often more convenient to specify the variable b_X since, for instance, the instantaneous Sharpe ratios are typically expressed in this variable. However, we argue that any trader would rather specify the variable μ_X. The reason, which we expand on in the next section, is that under mild regularity conditions on σ_X the sample average $\frac{1}{T}\int_0^T \mu_X(t)$ is an unbiased estimator of the continuously compounded rate of return $R_X^{cc}(T)$. Hence, from a trading perspective this is the natural view point, which we consequently adhere to. By specifying (μ_X, μ) rather than (b_X, b) the instantaneous Sharpe ratios take the form

$$s_X(t) = \frac{1}{2}\sigma_X(t) + \frac{\mu_X(t) - r(t)}{\sigma_X(t)}, \quad s_n(t) = \frac{1}{2}\sigma_n(t) + \frac{\mu_n(t) - r(t)}{\sigma_n(t)}, \quad 1 \leq n \leq N. \tag{17}$$

We end this section with a result covering the situation where the matrix V is only positive semi-definite.

Lemma 2.1. *Suppose that for all (t, ω) in some subset $\mathcal{A}$ of $[0, T] \times \Omega$, with positive product measure, there is a trading strategy w such that $\sigma_X(t) = 0$. Then*

$$\mu_X(t) = r(t), \quad \forall (t, \omega) \in \mathcal{A},$$

or there exists arbitrage opportunities.

Proof. It is a standard result that there can be only one locally risk-free asset in an arbitrage-free capital market, see for instance [7]. The assumed existence of a bank account, as in Eq. (5), concludes the proof. □

3 Kelly Trading

Let us consider an investor whose objective is to maximize the expected continuously compounded rate of return by trading in the available assets. Since

$$R_X^{cc}(T) = \frac{1}{T}\sum_{k=1}^{K}(T_k - T_{k-1})\, R_X^{cc}(T_k; T_{k-1}), \tag{18}$$

for any increasing partition $\{T_k\}$ such that $T_0 = 0$ and $T_K = T$, it is natural to conclude that the investor can equally maximize the logarithmic rate of return sequentially in time. More formally, it is well known that for any trading strategy w such that $\int_0^T \mathbb{E}\left[\sigma_X^2(t)\right] dt < \infty$, the expected rate of logarithmic return equals

$$\mathbb{E}\left[R_X^{cc}(T)\right] = \frac{1}{T}\int_0^T \mathbb{E}\left[\mu_X(t)\right] dt. \tag{19}$$

Hence, the logarithmic drift corresponds to the local rate of logarithmic return. It follows that we can maximize μ_X for each $\omega \in \Omega$, continuously in time, as shown below

$$\max_{\{w(s):0\leq s\leq T\}} \mathbb{E}\left[R_X^{cc}(T)\right] = \frac{1}{T}\int_0^T \max_{\{w(s):0\leq s\leq t\}} \mathbb{E}\left[\mu_X(t)\right] dt = \frac{1}{T}\int_0^T \mathbb{E}\left[\max_{w(t)} \mu_X(t)\right] dt. \tag{20}$$

We now turn to the expression for μ_X as given in Eq. (16) and notice, from the first order conditions, that

$$\hat{w}(t) = \arg\max_{w(t)} \mu_X(t) = V^{-1}(t)\,\sigma_{diag}(t)\, s(t) = \sigma_{diag}^{-1}(t)\,\rho^{-1}(t)\, s(t). \tag{21}$$

This trading strategy, called the optimal Kelly strategy after being popularized in [15], satisfies the fundamental equation of balance between return and volatility

$$\mu_X(t) - r(t) = \frac{1}{2}\sigma_X^2(t). \tag{22}$$

In order to appreciate the Kelly criterion it is highly informative to study the case of one single risky asset and a bank account. By the use of Eq. (11) and (16) we express the excess logarithmic return, for an arbitrary trading strategy, according to

$$\mu_X(t) - r(t) = w_1(t)\left(\mu_1(t) - r(t)\right) + \frac{1}{2}w_1(t)\left(1 - w_1(t)\right)\sigma_1^2(t). \tag{23}$$

On average the investor makes money in two different ways. Either through a directional bet on the excess logarithmic return of the risky asset or via the volatility. As long as the trader plays a fraction $0 < w_1 < 1$ of his wealth in the risky asset he makes money on volatility, where the most money to be maid is when $w_1 = 0.5$. Similarly, we see that any leveraged position $w_1 < 0$ or $w_1 > 1$ ends up loosing money on volatility. For instance, a short position in the risky asset always comes with a volatility drag. If, however, the risky asset has a non-zero excess logarithmic return it might still be optimal to leverage the position. The Kelly criterion $w_1 = 0.5 + \sigma_1^{-2}(\mu_1 - r)$ provides the right balance between directional and volatility trading. While, this simple formula changes as we introduce more assets it may serve as a rule of thumb to keep no less than half the wealth in risky assets when seeking optimal capital growth.

For many investors, though, such a strategy is deemed too risky in the sense that the potential return is not motivated by the high level of volatility needed to sustain the position. In the next result we use the optimal Kelly strategy to define what we henceforth call a Kelly strategy. However, before presenting the result we recall a standard result from linear algebra stating that any positive definite matrix A generates an inner product of the form $\langle u, v\rangle_A = u'Av$, with a corresponding norm $\|u\|_A^2 = u'Au$. The construction is a natural generalization of the Euclidean inner product, in which case A equals the identity matrix I.

Theorem 3.1. *Let $\hat{w}$ denote the optimal Kelly strategy that maximizes the logarithmic drift μ_X. Any trading strategy w that maximizes the magnitude of the instantaneous Sharpe ratio is of the form*

$$w(t) = k(t)\,\hat{w}(t) = k(t)\,\sigma_{diag}^{-1}(t)\,\rho^{-1}(t)\,s(t),$$

for some real-valued $\mathbb{F}$-adapted process k. We call such strategies Kelly strategies and we refer to the process k as the Kelly multiplier. The instantaneous Sharpe ratio of a Kelly strategy is independent of k and given by

$$\hat{s}_X^2(t) = s'(t)\,\rho^{-1}(t)\,s(t).$$

The corresponding logarithmic drift and volatility of such a trading strategy satisfy

$$\mu_X(t) = r(t) + \frac{1}{2}k(t)\,(2 - k(t))\,\hat{s}_X^2(t),$$
$$\sigma_X^2(t) = k^2(t)\,\hat{s}_X^2(t).$$

Proof. In terms of the inner product generated by the matrix V the instantaneous Sharpe ratio of the portfolio equals

$$s_X(t) = \frac{w'(t)\,\sigma_{diag}(t)\,s(t)}{\sqrt{w'(t)\,V(t)\,w(t)}} = \frac{\langle w(t), V^{-1}(t)\,\sigma_{diag}(t)\,s(t)\rangle_{V(t)}}{\|w(t)\|_{V(t)}} = \frac{\langle w(t), \hat{w}(t)\rangle_{V(t)}}{\|w(t)\|_{V(t)}}.$$

It now follows from Cauchy-Schwartz's inequality and Eq. (21) that

$$|s_X(t)| \leq \|\hat{w}(t)\|_{V(t)} = \sqrt{s'(t)\,\rho^{-1}(t)\,s(t)},$$

with equality if and only if w and $\hat{w}$ are collinear, that is $w = k\hat{w}$ for some $\mathbb{F}$-adapted real-valued process k. The proof concludes by using the expressions for μ_X and σ_X as in Eq. (11) and (16). □

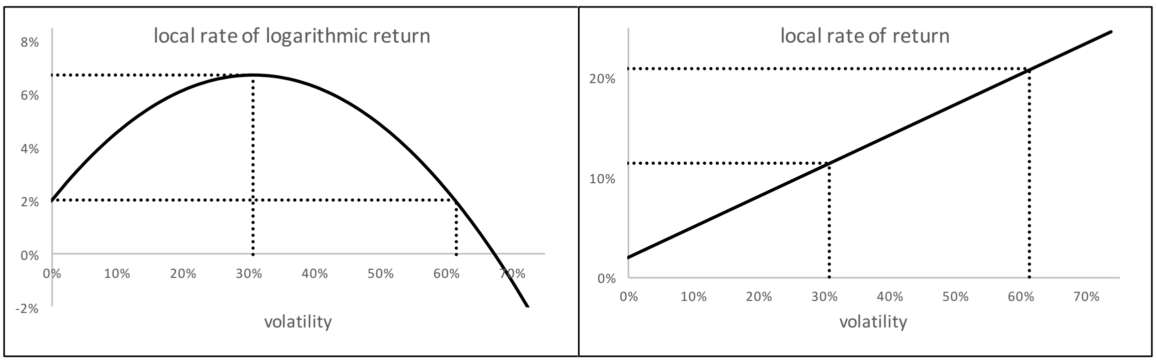

Figure 1: These plots show the local rate of logarithmic return μ_X and the local rate of return b_X as a function of the volatility σ_X. The underlying market parameters are: $r = 2\%$, $\mu = (3\%, 6\%)'$, $\sigma = (15\%, 25\%)'$ and $\rho_{1,2} = 10\%$. The dotted lines indicate the Kelly multiplier $k \in \{1, 2\}$.

The Kelly strategy derived in Theorem 3.1 allows us to control the risk via the Kelly multiplier k. For such a strategy we see that the logarithmic drift is positive for $k \in [0, 2]$ and negative otherwise. Moreover, the logarithmic drift is symmetric around its maximum value achieved for $k = 1$ and corresponding to the optimal Kelly strategy. Hence, the trading strategies $k = 1 \pm c$ yield the same logarithmic drift but have different volatilities. For this reason we see that it is never efficient to use a value of k outside of the interval $[0, 1]$. In the literature Kelly strategies are sometimes referred to as fractional Kelly strategies where fractional in this context relates to the optimal Kelly strategy and not to the portfolio wealth. In this paper, however, we exclusively use the term fractional in reference to the portfolio wealth. In Fig. 1 we plot a typical example of μ_X and b_X as a function of σ_X with the regions $k \in \{1, 2\}$ highlighted. While it is clear from the plot of μ_X that it is never efficient to use a value of k outside the interval $[0, 1]$ the plot of b_X reveals no such information. In other words, the instantaneous Sharpe ratio cannot be used to determine how much leverage an investor can accept. All that can be said is the following

Proposition 3.2. *Suppose there exists no arbitrage opportunities in the market. The instantaneous Sharpe ratio of a Kelly trader then satisfies*

$$\max_{1 \leq n \leq N} s_n^2(t) \leq \hat{s}_X^2(t) < \infty.$$

Proof. The lower bound follows directly from Theorem 3.1. In order to prove that an upper bound exists a.s. we consider the Kelly ratio $k = m/\hat{s}_X^2$ for some constant $m \in \mathbb{R}$. This yields

$$\mu_X(t) = r(t) + \frac{1}{2}\frac{m}{\hat{s}_X^2(t)}\left(2 - \frac{m}{\hat{s}_X^2(t)}\right)\hat{s}_X^2(t) \to r(t) + m, \quad \sigma_X^2(t) = \left(\frac{m}{\hat{s}_X^2(t)}\right)^2 \hat{s}_X^2(t) \to 0,$$

in the limit as $\hat{s}_X^2 \to \infty$. It now follows from Lemma 2.1 that there exists arbitrage opportunities if $\hat{s}_X^2 = \infty$, which concludes the proof by reversing the logic. □

At this point it is of great interest to make a comparison with the one-period mean-variance model of Markowitz [10] and the contributions by Tobin [11] and Merton [16]. In short, the main results states that the mean-variance optimal portfolio, consisting of risky assets only, yields a parabolic efficient frontier. Along the efficient frontier there is one and only one portfolio for which the Sharpe ratio is maximized. If we construct a new portfolio consisting of a bond and the assets-only maximal Sharpe ratio mean-variance portfolio the new efficient frontier becomes a straight line. Along the straight line the Sharpe ratio is preserved and, of course, equal to the maximal Sharpe ratio over the assets-only efficient frontier. Furthermore, the straight line coincides with the efficient frontier generated by a mean-variance optimization when the bond is added to the opportunity set of the risky assets. We illustrate the efficient frontiers in Fig 2 where we have used deterministic market parameters (r, μ, σ, ρ) to compute the expected value and the standard deviation of the periodic rate of return, see [12] for a simple account of the technical details. In Fig 2 we also include the Kelly region describing how much leverage an investor can take on. Stated differently, if the opportunity set of the mean-variance optimization contains both the bond and the risky assets the investor cannot use the Sharpe ratio to rank the portfolios along the straight line efficient frontier. Additional information about the risk appetite is needed to choose a particular level of mean and variance. In the Kelly framework the situation is in many ways similar but yet fundamentally different. First, we see from Eq. (12) that there is, by definition, a linear relationship between b_X and σ_X. As shown in Theorem 3.1, all Kelly strategies have the same instantaneous Sharpe ratio, which is the maximal instantaneous Sharpe ratio $\hat{s}_X$. However, different from the mean-variance analysis there is a unique portfolio that maximizes the drift of $\log(X)$ and thus the expected value of the continuously compounded rate of return. This portfolio is the one obtained by setting the Kelly multiplier $k = 1$.

In order to understand what drives the discrepancies between the two approaches one notices that the rate of return concept is not consistent. In the mean-variance framework the periodic rate of return is used while in the Kelly framework it is the continuously compounded rate of return that is applied. For this reason we consider the discretely compounded rate of return R_X^{dc} implicitly defined by

$$\frac{X(T)}{X(0)} = \left(1 + \frac{1}{f}R_X^{dc}(T)\right)^{fT}, \quad fT \geq 1, \tag{24}$$

where f denotes the frequency of compounding per year. This rate of return bridges the two concepts and by the use of Itô's lemma, on $X^{1/fT}$, we find that

$$R_X^{dc}(T) = \frac{1}{T}\int_0^T \left(\frac{X(t)}{X(0)}\right)^{\frac{1}{fT}} \left(\mu_X(t) + \frac{1}{2fT}\sigma_X^2(t)\right) dt + \frac{1}{T}\int_0^T \left(\frac{X(t)}{X(0)}\right)^{\frac{1}{fT}} \Sigma_X'(t)\, dW(t). \tag{25}$$

Hence, for any sufficiently regular trading strategy this leads to the optimization problem

$$\max_{\{w(s):0 \leq s \leq T\}} \mathbb{E}\left[R_X^{dc}(T)\right] = \frac{1}{T}\int_0^T \mathbb{E}\left[\left(\frac{X(t)}{X(0)}\right)^{\frac{1}{fT}} \max_{w(t)} \left(\mu_X(t) + \frac{1}{2fT}\sigma_X^2(t)\right)\right] dt, \tag{26}$$

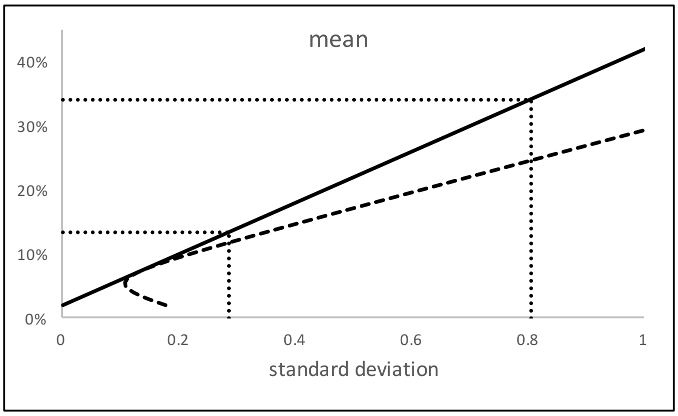

Figure 2: The plot shows typical efficient frontiers in the mean-variance framework when the opportunity set equals: bond and assets (solid line) and assets only (dotted line). That is, we plot the expected periodic rate of return $\mathbb{E}\left[R_X^p(T)\right]$ against its standard deviation. The market parameters are similar to those in Fig. 1 and the investment horizon T equals 2 years. The dotted lines indicate the Kelly multiplier $k \in \{1, 2\}$.

and from Eq. (11) and (16), it follows that the optimal trading strategy equals

$$\arg\max_{w(t)} \left(\mu_X(t) + \frac{1}{2fT}\sigma_X^2(t) \right) = \frac{fT}{fT-1}\sigma_{diag}^{-1}(t)\,\rho^{-1}(t)\,s(t) = \frac{fT}{fT-1}\hat{w}(t). \tag{27}$$

According to Theorem 3.1 the optimal trading strategy has a maximal instantaneous Sharpe ratio since it is of the form $k\hat{w}$, with $k = 1 + 1/(fT-1) \geq 1$. However, such a trading strategy should never be used for two reasons. First, if $fT \geq 2$ the trading strategy defined by $k = 1 - 1/(fT-1)$ yields the same logarithmic drift μ_X but at a lower volatility σ_X and second if $fT < 2$ the bank account outperforms the trading strategy in the sense that $r > \mu_X$. With this observation in mind we return to the one-period mean-variance framework where $fT = 1$. Even if continuous trading were allowed a strategy based on maximizing the periodic rate of return would imply a logarithmic drift $\mu_X = -\infty$ and an infinite volatility σ_X. Such a strategy would force the portfolio process X to zero at which point the investor would have no more money to play with. In other words, the investor would go bankrupt. To be mathematically precise when dealing with bankruptcy one should define the set of *admissible* trading strategies as those trading strategies for which $X \geq 0$. If, in the optimization problem, we restrict ourselves to admissible trading strategies it is easy to show that bankruptcy is an absorbing state, see Remark 3.3.4 in [7]. Of course, in a proper one-period model where only buy-hold strategies exist we face the far worse scenario of ending up in debt. In order to circumvent the problems associated with the situation $fT = 1$ the mean-variance approach of [10] either minimizes the variance given a constraint on the mean or equally maximizes the mean given a constraint on the variance. Below, we analyze the latter approach

$$\max_{w(t)} \left(\mu_X(t) + \frac{1}{2fT}\sigma_X^2(t) \right) \quad s.t. \quad \sigma_X^2(t) \leq \sigma_{target}^2(t). \tag{28}$$

By formulating the problem using a Lagrange multiplier

$$\max_{w(t)} \min_{\lambda \geq 0} \left(\mu_X(t) + \frac{1}{2fT}\sigma_X^2(t) + \lambda\left(\sigma_{target}^2(t) - \sigma_X^2(t)\right) \right), \tag{29}$$

and reversing the order of the max-min optimization we obtain, from Eq. (11) and (16), the first order conditions

$$\sigma_{diag}(t)\,s(t) - \left(1 - \frac{1}{fT} + 2\lambda\right) V(t)\,w(t) = 0, \quad w'(t)\,V(t)\,w(t) = \sigma_{target}^2(t). \tag{30}$$

Straightforward calculations then yield a solution expressed in terms of the optimal non-constraint Kelly strategy $\hat{w}$ and the corresponding maximal instantaneous Sharpe ratio $\hat{s}_X$ as shown below

$$w(t) = \frac{\sigma_{target}(t)}{\hat{s}_X(t)}\hat{w}(t), \quad \sigma_{target}(t) \geq 0. \tag{31}$$

Hence, by imposing a constraint on the instantaneous volatility we can maximize the expectation of the discretely compounded rate of return in such a way that the resulting optimal trading strategy is independent of the compounding frequency f. For any value of σ_{target} the resulting portfolio has a maximal instantaneous Sharpe ratio given by $\hat{s}_X$ as proved in Theorem 3.1. However, the main issue remains in the sense that for target volatilities $\sigma_{target} > \hat{s}_X$ there is always a better trading strategy to choose which generates the same logarithmic drift μ_X but at a lower volatility σ_X. Moreover, for $\sigma_{target} > 2\hat{s}_X$ the bank account starts to outperform the portfolio, which implies an increasing probability for the investor to go bankrupt.

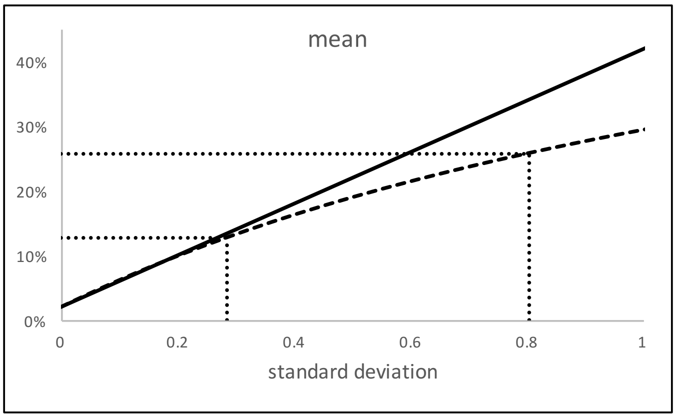

Figure 3: The plot shows the expected periodic rate of return $\mathbb{E}\left[R_X^p\left(T\right)\right]$ against its standard deviation, when the opportunity set equals bond and risky assets, for: mean-variance portfolio (solid line) and Kelly portfolio (dotted line). The market parameters are similar to those in Fig. 1 and the investment horizon T equals 2 years. The dotted lines indicate the Kelly multiplier $k \in \{1, 2\}$.

In Fig. 3 we compare the mean-variance buy-hold strategy with the Kelly strategy by plotting the mean and the standard deviation of the periodic return. The plot is generated under the assumption of deterministic market parameters (r, μ, σ, ρ). It follows that the straight line efficient frontier corresponding to a mean-variance optimization over the bond and the risky assets is associated with high probability of the investor ending up in debt whenever $\sigma_{target} > 2\hat{s}_X$ and most likely much earlier since $\sigma_{target} > \hat{s}_X$ already implies that the corresponding Kelly strategy is inefficient. Note that any mean-variance buy-hold strategy comes with a positive probability to end up in debt if the trading strategy is leveraged. The Kelly portfolio accurately shows that if we want to double the excess rate of return it is not enough to enter a leveraged position with double the standard deviation. This observation highlights the difference between multi-period and one-period optimization problems. It is naive to think that a multi-period optimization problem can be treated as a sequence of independent one-period optimization problems unless we add constraints prevailing the investor to go bankrupt or, worse, end up in debt.

4 Utility representation of the Kelly Strategy

It is often claimed that the optimal Kelly strategy is equivalent to that of an investor having a logarithmic utility function and thus does not provide any new insights into how best to choose the trading strategy. At the same time, one sometimes hears statements arguing that it is a mere coincident that the optimal Kelly strategy agrees with that of a logarithmic utility maximizing investor. In this section we try to shed some light on these opposing views.

The von Neumann-Morgenstern expected utility representation, see [18], guarantees the existence of a function $u : \mathbb{R}_+ \to \mathbb{R}$, which we generally assume to be strictly increasing and strictly concave such that it can be regarded as the utility function of a risk averse individual. By invoking the Itô formula and imposing sufficient regularity conditions the expectation of the diffusion term vanishes, leaving us with

$$\mathbb{E}\left[u\left(X\left(T\right)\right)\right] = u\left(X\left(0\right)\right) + \int_0^T \mathbb{E}\left[u'\left(X\left(t\right)\right)X\left(t\right)\left(\mu_X\left(t\right) + \frac{1}{2}\left(1 + \frac{u''\left(X\left(t\right)\right)X\left(t\right)}{u'\left(X\left(t\right)\right)}\right)\sigma_X^2\left(t\right)\right)\right]dt. \quad (32)$$

If, in addition to the assumption $u'\left(X\right)X > 0$, we let p denote the relative risk aversion corresponding to u it follows that the optimal trading strategy takes the form

$$\underset{w(t)}{\arg\max}\left(\mu_X\left(t\right) + \frac{1}{2}\left(1 - p\left(X\left(t\right)\right)\right)\sigma_X^2\left(t\right)\right) = \frac{1}{p\left(X\left(t\right)\right)}\hat{w}\left(t\right). \quad (33)$$

A comparison with Theorem 3.1 then shows that this is an instantaneous Sharpe maximizing trading strategy of the form $w = k\hat{w}$. Hence, the general von Neumann-Morgenstern expected utility framework is in fact embedded in the

Kelly framework since we can always choose the Kelly multiplier to equal the reciprocal of the relative risk aversion coefficient for any strictly increasing and strictly concave utility function. In the particular case where p is constant the corresponding utility function is unique, up to affine transformations, and given by the power utility function

$$u(x) = \frac{x^{1-p} - 1}{1-p}, \quad p > 0, \tag{34}$$

which includes the logarithmic utility function for $p = 1$. Note, however, that this does not imply that the Kelly framework is a special case of the expected utility framework. What it tells us is that the optimal Kelly strategy and some other Kelly strategies, like the ones with constant Kelly multiplier $k > 0$, are also contained in the expected utility framework. Hence, even though the expected utility framework is a subset of the Kelly framework the main insight from the Kelly approach is completely lost in the von Neumann-Morgenstern framework, namely that it is never efficient to use a multiplier k outside of the interval $[0, 1]$. In other words, risk averse individuals are unlikely to be rational unless the relative risk aversion coefficient $p \geq 1$. Over the years there has been a fierce debate between the camps of expected utility and growth supporters, see for instance Samuelson [14] and Hakansson [5] for a historical, yet very relevant, account.

While addressing the expected utility approach we should mention that also the mean-variance framework of Markowitz and Tobin can be represented by a utility function. In this case, however, the utility function is known to be quadratic and therefore not increasing. Hakansson [4] and [5] argued early that the optimal growth theory had far better theoretical foundations and thus deserved more attention than was given. Looking back at the past we see that this did not happen, the mean-variance approach has completely dominated the debate despite its obvious short-comings. Additionally, the expected utility approach has been favoured against the optimal growth theory. We claim that these have been unwise decisions. The Kelly approach answers the important question of how much leverage an investor can take on, which is missing in both the mean-variance and the expected utility approach. Furthermore, as we show in the next section, the Kelly approach agrees with the mean-variance approach on fundamental separation results.

5 Separation of Investment Decisions

In this section we show that a Kelly trader can separate his investment decisions in a way similar to what takes place in the mean-variance framework. More notably we show that the Kelly trader allocates part of his wealth in a particular assets-only mutual fund and the remaining wealth in the bank account. In order to launch this study we introduce some primary trading strategies in addition to the reference strategy $u^0 = (0, \ldots, 0)'$ corresponding to the bank account. We express these primary trading strategies in terms of b rather than μ for convenience only and recall that μ is the quantity specified by the investors.

Lemma 5.1. *Consider the trading strategies*

$$u^1(t) = V^{-1}(t)\left(b(t) - b_{\min}(t)\,\mathbf{1}_N\right), \quad u^2(t) = \sigma^2_{\min}(t)\,V^{-1}(t)\,\mathbf{1}_N,$$

where

$$b_{\min}(t) = \sigma^2_{\min}(t)\,\langle b(t), \mathbf{1}_N\rangle_{V^{-1}(t)}, \quad \sigma_{\min}(t) = \|\mathbf{1}_N\|^{-1}_{V^{-1}(t)}.$$

Then

$$\mathbf{1}'_N u^1(t) = 0, \quad \mathbf{1}'_N u^2(t) = 1, \quad \langle u^1, u^2\rangle_{V(t)} = 0.$$

Moreover, if we define

$$\sigma^2_0(t) = \|b(t)\|^2_{V^{-1}(t)} - \left(\frac{b_{\min}(t)}{\sigma_{\min}(t)}\right)^2,$$

the local characteristics of the trading strategies, indicated as subscripts, equal

$$\mu_{u^1}(t) = b_{u^1}(t) - \frac{1}{2}\sigma^2_{u^1}(t), \quad b_{u^1}(t) = r(t) + \sigma^2_0(t), \quad \sigma^2_{u^1}(t) = \sigma^2_0(t),$$
$$\mu_{u^2}(t) = b_{u^2}(t) - \frac{1}{2}\sigma^2_{u^2}(t), \quad b_{u^2}(t) = b_{\min}(t), \quad \sigma^2_{u^2}(t) = \sigma^2_{\min}(t).$$

Proof. The first part of the proof follows from straightforward calculations and is thus omitted. For the second part it follows from Cauchy-Schwartz's inequality that $\sigma^2_0 \geq 0$. Hence, σ_0 is well defined. To conclude the proof we apply the explicit expressions for the trading strategies to Eqs. (10) and (11). □

We regard the portfolios associated with the trading strategies $\left(u^1, u^2\right)$ as mutual funds directly available for investments. It follows that the strategy u^1 has full exposure to the bank account and is market neutral in the risky assets, while u^2 is the assets-only trading strategy with minimal variance, see [12] for details. We further assume, without loss of generality, that $b_{\min} > r$ since otherwise the bank account is clearly preferable. Throughout this paper we let v denote an $\mathbb{R}^2$-valued $\mathbb{F}$-predictable self-financing trading strategy taking positions in two mutual funds as opposed to the $\mathbb{R}^N$-valued self-financing trading strategy w which takes positions in each of the risky assets $P = (P_1, \ldots, P_N)'$. We say that a trading strategy w has a mutual fund representation if $w = v_1 u^1 + v_2 u^2$. Furthermore, such a trading strategy is self-financing if

$$\frac{dX_w(t)}{X_w(t)} = (1 - v_1(t) - v_2(t)) \frac{dX_{u^0}(t)}{X_{u^0}(t)} + v_1(t) \frac{dX_{u^1}(t)}{X_{u^1}(t)} + v_2(t) \frac{dX_{u^2}(t)}{X_{u^2}(t)}, \tag{35}$$

with the subscript indicating the corresponding mutual fund. It follows from Lemma 5.1 that the total fraction invested in the bank account (adding the contributions from each mutual fund) equals

$$w_B(t) = 1 - \mathbf{1}_N' w(t) = 1 - v_2(t), \tag{36}$$

such that an assets-only portfolio must have $v_2 = 1$. We are now ready to give the mutual fund version of Theorem 3.1.

Theorem 5.2. *Let the mutual funds $\left(u^1, u^2\right)$ be given as in Lemma 5.1 and let $\hat{w}$ denote the optimal Kelly strategy that maximizes the logarithmic drift μ_X. Any self-financing trading strategy of the form $w = v_1 u^1 + v_2 u^2$ that maximizes the magnitude of the instantaneous Sharpe ratio must satisfy*

$$w(t) = v_1(t) \left(u^1(t) + \frac{b_{\min}(t) - r(t)}{\sigma^2_{\min}(t)} u^2(t) \right) = v_1(t) \hat{w}(t).$$

The instantaneous Sharpe ratio is independent of v_1 and given by

$$\hat{s}^2_X(t) = s^2_{u^1}(t) + s^2_{u^2}(t) = \sigma^2_0(t) + \left(\frac{b_{\min}(t) - r(t)}{\sigma_{\min}(t)} \right)^2.$$

The corresponding logarithmic drift and volatility of such a trading strategy satisfy

$$\mu_X(t) = r(t) + \frac{1}{2} v_1(t) (2 - v_1(t)) \hat{s}^2_X(t),$$
$$\sigma^2_X(t) = v_1^2(t) \hat{s}^2_X(t).$$

Proof. We use Lemma 5.1 to calculate the local rate of return and volatility processes for an arbitrary trading strategy of the form $w = v_1 u^1 + v_2 u^2$. This yields

$$b_X(t) = r(t) + v_1(t) \sigma^2_0(t) + v_2(t) (b_{\min}(t) - r(t)), \quad \sigma^2_X(t) = v_1^2(t) \sigma^2_0(t) + v_2^2(t) \sigma^2_{\min}(t).$$

Straightforward calculations now show that the square of the instantaneous Sharpe ratio

$$s^2_X(t) = \frac{\left(v_1(t) \sigma^2_0(t) + v_2(t) (b_{\min}(t) - r(t))\right)^2}{v_1^2(t) \sigma^2_0(t) + v_2^2(t) \sigma^2_{\min}(t)} = \frac{\left(\sigma^2_0(t) + \left(\frac{v_2(t)}{v_1(t)}\right) (b_{\min}(t) - r(t))\right)^2}{\sigma^2_0(t) + \left(\frac{v_2(t)}{v_1(t)}\right)^2 \sigma^2_{\min}(t)},$$

is maximized along the line

$$\frac{v_2(t)}{v_1(t)} = \frac{b_{\min}(t) - r(t)}{\sigma^2_{\min}(t)}.$$

The proof concludes by evaluating all the terms along the optimal line. □

It is interesting to note that the optimal squared instantaneous Sharpe ratio can be expressed as a sum of squared instantaneous mutual fund Sharpe ratios. It would be hard to conjuncture such a result from Theorem 3.1. We now turn our focus to mutual funds consisting of risky assets only. As seen from Eq. (36) this implies that v_2 must be set equal to one.

Corollary 5.3. *Among all self-financing trading strategies of the form $w = v_1 u^1 + u^2$ the assets-only instantaneous efficient frontier is characterized by*

$$b_X(t) = v_1(t) \sigma^2_0(t) + b_{\min}(t), \quad \sigma^2_X(t) = v_1^2(t) \sigma^2_0(t) + \sigma^2_{\min}(t) \geq \sigma^2_{\min}(t).$$

Under the assumption that the interest rate $r < b_{\min}$ there exists a unique assets-only portfolio with maximal instantaneous Sharpe ratio. This portfolio is given by

$$u^3(t) = \frac{\sigma^2_{\min}(t)}{b_{\min}(t) - r(t)} \hat{w}(t).$$

Furthermore, the mutual fund u^3 is Kelly optimal if the interest rate r equals $r^* = b_{\min} - \sigma^2_{\min}$.

Proof. The results follow directly from the proof of Theorem 5.2. □

We note that the assets-only instantaneous efficient frontier describes a parabolic curve with a unique point of minimum volatility $\sigma_{\min}$. The local rate of return associated with this level $b_{\min}$ is further minimal over the efficient branch corresponding to $v_1 \geq 0$. However, since $b_{\min}$ does not represent a true local minimum we often use the term $r^* + \sigma^2_{\min}$ instead.

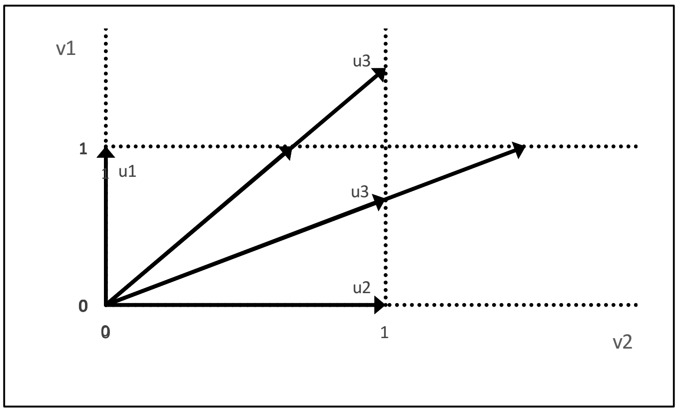

Figure 4: The plot shows the relation between the optimal Kelly strategy and the collinear assets-only mutual fund u^3. All optimal Kelly strategies are located along the line $v_1 = 1$ where the realized point depends on the level of the interest rate r, while all assets-only strategies are located along the line $v_2 = 1$. The point $v_1 = 1, v_2 = 0$ corresponds to $r = r^* + \sigma^2_{\min}$ and $v_1 = 1, v_2 = 1$ corresponds to $r = r^*$. Consequently, the collinear mutual fund u^3 is a Kelly strategy with: inefficient Kelly multiplier, $k > 1$, if $r^* < r \leq r^* + \sigma^2_{\min}$ and efficient Kelly multiplier, $k < 1$, if $r \leq r^*$.

The assets-only mutual fund u^3 in Corollary 5.3 is seen to be a Kelly strategy which can be more risky than the optimal Kelly strategy $\hat{w}$ depending on the level of interest rate r. We highlight this observation in Fig. 4. While all assets-only portfolios lie on the line $v_2 = 1$, all optimal Kelly strategies lie on the line $v_1 = 1$. It is the realized interest rate r that causes the optimal Kelly strategy to move. For instance, the point $v_1 = 1, v_2 = 0$ corresponds to the interest rate $r = b_{\min}$ while $v_1 = 1, v_2 = 1$ corresponds to $r = r^*$. It follows that the Kelly strategy u^3 is efficient if $r \leq r^*$ and inefficient if the interest rate is in the interval $\left(r^*, r^* + \sigma^2_{\min}\right)$. Whichever the situation a Kelly trader can generate any trading strategy of the form $k\hat{w}$, as analyzed in Theorems 3.1 and 5.2, by trading in the assets-only mutual fund u^3 and in the bank account. In other words, such a Kelly trader modifies the risk profile of the mutual fund u^3 to realize a desired Kelly multiplier according to

$$\frac{dX_{k\hat{w}}(t)}{X_{k\hat{w}}(t)} = v(t)\frac{dX_{u^3}(t)}{X_{u^3}(t)} + (1 - v(t))\frac{dB(t)}{B(t)}, \quad v(t) = k(t)\left(1 - \frac{r(t) - r^*(t)}{\sigma^2_{\min}(t)}\right). \tag{37}$$

Summing up we have shown that on the assets-only instantaneous efficient frontier there exists a trading strategy u^3, with maximal instantaneous Sharpe ratio, that is collinear with any Kelly strategy. This proves that separation of investment decisions works much the same way in the Kelly framework as it does in the mean-variance framework. That is, an investor can achieve an optimal instantaneous Sharpe allocation by simply taking positions in the risk-free asset and in a particular mutual fund. The new efficient frontier, constructed in this way, becomes a straight line and every allocation on the line has the same instantaneous Sharpe ratio. What is different between the two approaches, though, is that in the Kelly framework the optimal Kelly strategy $\hat{w}$, which maximizes the logarithmic rate of return, lies on the efficient frontier. Such a result is in general impossible to have in the mean-variance framework where only buy-hold strategies are allowed. The reason is that the terminal value of the portfolio can become negative in which case the logarithmic return cannot even be defined. In the Kelly framework, by contrast, it is the ability to continuously trade the assets that ensures the portfolio process to remain positive.

6 Equilibrium in the Covariance Market

If we consider a market consisting of a number of investors we can expect that their joint actions of trading will force the market to some kind of equilibrium. In this section we make precise what equilibrium means in terms of the covariance matrix V. We assume that each investor tries to maximize the continuously compounded rate of return corresponding to his portfolio. Hence, the performance of each investor can be measured objectively. This allows us to formulate the equilibrium as the solution to a reduced two-person zero sum differential game and apply the min-max concept of von Neumann [17]. It is well-known that any equilibrium found in this way is also a Nash equilibrium. We stress the importance of an objective performance criteria and argue that it would not have been possible to characterize the market equilibrium in this way had we assumed the investors to be expected utility maximizers. The reason is, as explained earlier, that the associated utility function is only unique up to affine transformations. The approach taken below is to consider particular sub-problems and thereafter piecing the results together.

6.1 One risky asset and bank account game

Let us consider an optimal Kelly trader who only invests in the bank account and one risky asset, say P_n, with instantaneous Sharpe ratio given by s_n. The associate trading strategy w_n is then real-valued and according to Theorem 3.1 we have

$$\max_{w_n(t)} \mu_X(t) = r(t) + \frac{1}{2} s_n^2(t), \quad \arg\max_{w_n(t)} \mu_X(t) = \frac{s_n(t)}{\sigma_n(t)}, \tag{38}$$

where the instantaneous Sharpe ratio, expressed in terms of the logarithmic drift μ_n, takes the form

$$s_n(t) = \frac{1}{2}\sigma_n(t) + \frac{\mu_n(t) - r(t)}{\sigma_n(t)}. \tag{39}$$

Without going into details at this stage we claim that trading in an asset affects the price through the volatility of the asset. What one trader makes in profit is a potential loss for another and the role of the market is to establish the correct level of volatility. This zero-sum competition forces the prices towards an equilibrium and here we apply the min-max formulation

$$\min_{\sigma_n(t)} \max_{w_n(t)} \mu_X(t) = \max_{w_n(t)} \min_{\sigma_n(t)} \mu_X(t). \tag{40}$$

Hence, rather than letting every trader trying to optimize his logarithmic drift (or equally his continuously compounded rate of return) we take the view point that all traders but our optimal Kelly trader gang up and try to minimize the Kelly trader's potential profit. From the first order conditions $s_n \partial s_n / \partial \sigma_n = 0$ it then follows that

$$\frac{s_n(t)}{\sigma_n(t)} (\sigma_n(t) - s_n(t)) = 0. \tag{41}$$

The solutions ($s = 0$ and $s = \sigma$) to this equation can be summarized by

$$\sigma_n^*(t) = \sqrt{2\,|\mu_n(t) - r(t)|}. \tag{42}$$

Hence, the market sets the volatility such that each risky asset satisfies the fundamental equation of optimal balance between return and volatility

$$|\mu_n(t) - r(t)| = \frac{1}{2}(\sigma_n^*(t))^2, \tag{43}$$

in much the same way as in Eq. (22). We verify that the solution corresponds to a minimum by computing the second derivative

$$\frac{\partial^2}{\partial \sigma_n^2} \max_{w_n(t)} \mu_X(t)\Big|_{\sigma_n(t)=\sigma_n^*(t)} = \left(\frac{\partial s_n}{\partial \sigma_n}(t)\right)^2 + s_n(t)\frac{\partial^2 s_n}{\partial \sigma_n^2}(t)\Big|_{\sigma_n(t)=\sigma_n^*(t)} = 1 > 0. \tag{44}$$

This proves that

$$\min_{\sigma_n(t)} \max_{w_n(t)} \mu_X(t) = r(t) + (\mu_n(t) - r(t))_+, \tag{45}$$

and that the optimal trading strategy equals

$$w_n^*(t) = e_n(t), \quad e_n(t) = \frac{1}{2}(1 + \operatorname{sgn}(\mu_n(t) - r(t))) = \begin{cases} 1, & \mu_n(t) > r(t) \\ \frac{1}{2}, & \mu_n(t) = r(t) \\ 0, & \mu_n(t) < r(t) \end{cases}. \tag{46}$$

It is interesting to note that even if the Kelly investor plays optimally the other market participants gang up on him in such a way that he can basically do no better than to go locally buy-hold in either the risky asset or the bank account. Only when $\mu_n = r$ will a mixed strategy be used. Below we show that this is indeed the equilibrium point of the market.

Theorem 6.1. *Given an optimal Kelly trader who only invests in the bank account and one risky asset, say P_n. If the capital market is arbitrage-free then*

$$\min_{\sigma_n(t)} \max_{w_n(t)} \mu_X(t) = \max_{w_n(t)} \min_{\sigma_n(t)} \mu_X(t) = r(t) + (\mu_n(t) - r(t))_+.$$

The equilibrium solution (σ_n^*, w_n^*) is given by Eq. (42) and (46), respectively.

Proof. Since we have already analyzed the situation when the market participants gang up on the optimal Kelly trader it remains to consider the response of the Kelly trader to the market coalition.

Let μ_X be given by Eq. (16) such that the first order condition corresponding to the market coalition's optimization problem $\min_{\sigma_n} \mu_X$ is: $w_n (1 - w_n) \sigma_n = 0$. Hence, the market coalition will force σ_n^* to zero if $w_n \notin \{0, 1\}$. Eq. (11) now shows that $\sigma_X = 0$ for all trading strategies $w_n \neq 1$ and by the use of Lemma 2.1 we conclude that $\mu_X = r$ if $w_n \neq 1$. Therefore

$$\min_{\sigma_n(t)} \mu_X(t) = r(t) + (\mu_n(t) - r(t)) \mathbf{1}\{w_n(t) = 1\}.$$

An optimal Kelly trader facing such an objective function can do no better than to set $w_n = 1$ if $\mu_n > r$. Hence, while the equilibrium pair (σ_n^*, w_n^*) is uniquely defined for the problem $\min_{\sigma_n} \max_{w_n} \mu_X$ it is one of many solutions to the problem $\max_{w_n} \min_{\sigma_n} \mu_X$. □

Given the asset volatility σ_n we know from Theorem 3.1 that there exists a trading strategy w_n such that $\mu_X = r + \frac{1}{2} s_n^2$ and $\sigma_X^2 = s_n^2$. The trading strategy is the optimal Kelly strategy defined by $\hat{w}_n = s_n / \sigma_n$. Below we show that there exist a dual volatility $\tilde{\sigma}_n$ and a dual trading strategy $\tilde{w}_n$ that generates the same logarithmic drift μ_X and volatility σ_X. The dual pair is related to the equilibrium according to

$$\tilde{\sigma}_n(t) = \frac{(\sigma_n^*(t))^2}{\sigma_n(t)}, \quad \tilde{w}_n(t) = \hat{w}_n(t) \left(\frac{\sigma_n(t)}{\tilde{\sigma}_n(t)} \right) (2e_n(t) - 1). \tag{47}$$

In order to verify the construction we observe that

$$\frac{\tilde{s}_n(t)}{s_n(t)} = \frac{\tilde{w}_n(t) \tilde{\sigma}_n(t)}{\hat{w}_n(t) \sigma_n(t)} = 2e_n(t) - 1 = \begin{cases} +1, & \mu_n(t) > r(t) \\ 0, & \mu_n(t) = r(t) \\ -1, & \mu_n(t) < r(t) \end{cases}. \tag{48}$$

Hence, $|\tilde{s}_n| = |s_n|$ which shows that μ_X and σ_X, corresponding to the optimal Kelly strategy, are invariant. We now extend the analysis to arbitrary trading strategies.

Corollary 6.2. *Given a risky asset P_n with volatility σ_n. Let w_n be an arbitrary trading strategy with logarithmic drift μ_X and volatility σ_X. There exist a dual volatility $\tilde{\sigma}_n$ and a dual trading strategy $\tilde{w}_n$ that generate the same logarithmic drift and volatility. The dual pair is given by*

$$\tilde{\sigma}_n(t) = \frac{(\sigma_n^*(t))^2}{\sigma_n(t)}, \quad \tilde{w}_n(t) = w_n(t) \left(\frac{\sigma_n(t)}{\tilde{\sigma}_n(t)} \right) (2e_n(t) - 1),$$

where σ_n^* denotes the equilibrium volatility and $w_n^* = e_n$ the equilibrium trading strategy.

Proof. Let us first observe that $|2e_n - 1| \sigma_n^* = \sigma_n^*$. From the definition of the equilibrium volatility σ_n^* we then have

$$s_n(t) = \frac{1}{2} \sigma_n^*(t) \left(\frac{\sigma_n(t)}{\sigma_n^*(t)} + (2e_n(t) - 1) \frac{\sigma_n^*(t)}{\sigma_n(t)} \right), \quad \tilde{s}_n(t) = \frac{1}{2} \sigma_n^*(t) \left(\frac{\tilde{\sigma}_n(t)}{\sigma_n^*(t)} + (2e_n(t) - 1) \frac{\sigma_n^*(t)}{\tilde{\sigma}_n(t)} \right),$$

such that by using the the dual volatility $\tilde{\sigma}_n$ we obtain

$$(2e_n(t) - 1) \tilde{s}_n(t) = s_n(t), \quad (2e_n(t) - 1) \left(\frac{\sigma_n(t)}{\sigma_n^*(t)} \right)^2 \tilde{\sigma}_n(t) \tilde{s}_n(t) = \sigma_n(t) s_n(t).$$

In order to conclude the proof we introduce analogue deterministic formulations of Eq. (11) and (16). That is

$$\sigma_X(w_n) = |w_n| \sigma_n, \quad \tilde{\sigma}_X(w_n) = |w_n| \tilde{\sigma}_n,$$
$$\mu_X(w_n) = r + w_n \sigma_n s_n - \frac{1}{2} \sigma_X^2(w_n), \quad \tilde{\mu}_X(w_n) = r + w_n \tilde{\sigma}_n \tilde{s}_n - \frac{1}{2} \tilde{\sigma}_X^2(w_n).$$

Hence, with $\xi_n = \sigma_n / \sigma_n^*$, we see that $\sigma_X(w_n) = \tilde{\sigma}_X\left(w_n (2e_n - 1) \xi_n^2\right)$ and $\mu_X(w_n) = \tilde{\mu}_X\left(w_n (2e_n - 1) \xi_n^2\right)$, for all $w_n \in \mathbb{R}$. □

We return to the dual volatility when studying the dynamic stability of the market. But first we must define the concepts of equilibrium correlation and dual correlation. However, before embarking on this analysis, let us stress one important observation from the proof above; namely that in equilibrium the instantaneous Sharpe ratio takes the simple form

$$s_n^*(t) = \sigma_n^*(t) e_n(t) = \begin{cases} \sigma_n^*(t), & \mu_n(t) > r(t) \\ 0, & \mu_n(t) \leq r(t) \end{cases}, \tag{49}$$

highlighting the fact that in equilibrium it is optimal to run an instantaneous buy-hold strategy in either the risky asset or the bank account.

6.2 Two risky assets and bank account game

Here we consider an optimal Kelly trader who only invests in the bank account and two risky assets, say P_n and P_m. Hence, the self-financing trading strategy $w = (w_n, w_m)'$ takes values in $\mathbb{R}^2$ and we associate the first component with P_n and the second component with P_m. By the use of Theorem 3.1 we find that the optimal Kelly strategy is given by

$$\arg\max_{w(t)} \mu_X(t) = \frac{1}{1-\rho_{n,m}^2(t)} \left(\frac{s_n(t) - \rho_{n,m}(t) s_m(t)}{\sigma_n(t)}, \frac{s_m(t) - \rho_{n,m}(t) s_n(t)}{\sigma_m(t)} \right)', \tag{50}$$

where (s_n, s_m) are the asset specific instantaneous Sharpe from Eq. (39) while

$$\rho_{n,m}(t) = \frac{\Sigma_n'(t)\Sigma_m(t)}{\sigma_n(t)\sigma_m(t)}. \tag{51}$$

The instantaneous Sharpe ratio of the optimal Kelly strategy equals

$$\hat{s}_X^2(t) = \frac{1}{1-\rho_{n,m}^2(t)} \left(s_n^2(t) - 2 s_n(t) \rho_{n,m}(t) s_m(t) + s_m^2(t) \right), \tag{52}$$

such that the optimal solution $\max_w \mu_X = r + \frac{1}{2}\hat{s}_X^2$. If, in addition, the market participants form a coalition against the optimal Kelly trader the resulting outcome is

$$\min_{\rho_{n,m}(t)} \max_{w(t)} \mu_X(t) = \min_{\rho_{n,m}(t)} \left(r(t) + \frac{1}{2}\hat{s}_X^2(t) \right). \tag{53}$$

The associated first order condition is given by the equation $\rho_{n,m} = s_n s_m / \hat{s}_X^2$, which has a unique solution under the constraint $|\rho_{n,m}| \leq 1$. The solution is given by

$$\rho_{n,m}^*(t) = \begin{cases} s_m(t)/s_n(t), & s_n^2(t) \geq s_m^2(t) \\ s_n(t)/s_m(t), & s_m^2(t) > s_n^2(t) \end{cases}. \tag{54}$$

Moreover

$$\left. \frac{\partial^2}{\partial \rho_{n,m}^2} \max_{w(t)} \mu_X(t) \right|_{\rho_{n,m}(t)=\rho_{n,m}^*(t)} = \left. \frac{1}{1-\rho_{n,m}^2(t)} \hat{s}_X^2(t) \right|_{\rho_{n,m}(t)=\rho_{n,m}^*(t)} \geq 0, \tag{55}$$

which proves that $\rho_{n,m}^*$ minimizes $\max_w \mu_X$. Straightforward algebraic manipulations, taking extra care of the degenerate case for which $|s_n| = |s_m|$, then verifies that the general solution can be expressed as

$$\min_{\rho_{n,m}(t)} \max_{w(t)} \mu_X(t) = r(t) + \frac{1}{2} \max\left(s_n^2(t), s_m^2(t) \right), \tag{56}$$

$$w_n^*(t) = \frac{s_n(t)}{\sigma_n(t)} e_{n,m}(t), \quad w_m^*(t) = \frac{s_m(t)}{\sigma_m(t)} \left(1 - e_{n,m}(t)\right), \tag{57}$$

with

$$e_{n,m}(t) = \frac{1}{2}\left(1 + \operatorname{sgn}\left(s_n^2(t) - s_m^2(t)\right)\right) = \begin{cases} 1, & s_n^2(t) > s_m^2(t) \\ \frac{1}{2}, & s_n^2(t) = s_m^2(t) \\ 0, & s_n^2(t) < s_m^2(t) \end{cases}. \tag{58}$$

This shows that the market coalition sets the equilibrium correlation in such a way that the two risky assets and bank account game reduces to a one risky asset and bank account game. Below we show that the pair $\left(\rho_{n,m}^*, w^*\right)$ results in a point of equilibrium.

Theorem 6.3. *Given a Kelly trader who only invests in the bank account and two risky assets, say P_n and P_m. If the capital market is arbitrage-free then*

$$\min_{\rho_{n,m}(t)} \max_{w(t)} \mu_X(t) = \max_{w(t)} \min_{\rho_{n,m}(t)} \mu_X(t) = r(t) + \frac{1}{2}\max\left(s_n^2(t), s_m^2(t)\right).$$

The equilibrium solution $\left(\rho_{n,m}^*, w^*\right)$ is given by Eq. (54) and (57), respectively.

Proof. Since we have already analyzed the situation when the market participants gang up on an optimal Kelly trader it remains to consider the response of the Kelly trader to the market coalition.

We first notice that if $\sigma_n \sigma_m = 0$ then at least one of the assets is indistinguishable from the bank account and therefore Proposition 2.1 can be applied instead. Hence, without loss of generality we may assume that $\sigma_n, \sigma_m > 0$.

Let μ_X be given by Eq. (16) such that the first order condition corresponding to the market coalition's optimization problem $\min_{\rho_{n,m}} \mu_X$ is: $w_n w_m \sigma_n \sigma_m = 0$. Hence, the market coalition threatens to force $\sigma_n \sigma_m$ to zero if the Kelly trader does not commit to a trading strategy such that $w_n w_m = 0$. An optimal Kelly trader facing such an objective function can do no better than to trade in the bank account and the asset with the highest instantaneous Sharpe ratio. □

If we further assume that the volatility market is in equilibrium, that is $\sigma_n = \sigma_n^*$ and $\sigma_m = \sigma_m^*$, it follows from Eq. (49) that

$$\min_{\rho_{n,m}(t)} \max_{w(t)} \mu_X (t) = r (t) + \max \left((\mu_n (t) - r (t))_+ , (\mu_m (t) - r (t))_+ \right) . \tag{59}$$

Hence, the market participants will basically force the optimal Kelly trader to take a local buy-hold position in either the bank account or the risky asset with the largest magnitude of the instantaneous Sharpe ratio.

Having derived the equilibrium correlation $\rho_{n,m}^*$ we now investigate whether we can specify a dual correlation $\tilde{\rho}_{n,m}$ that leaves the magnitude of the optimal instantaneous Sharpe ratio unaltered.

Corollary 6.4. *Given two risky asset P_n and P_m with instantaneous Sharpe ratios s_n and s_m and instantaneous asset-asset correlation $\rho_{n,m}$. Let $\hat{w} = (\hat{w}_n, \hat{w}_m)'$ be the the optimal Kelly strategy with instantaneous Sharpe ratio $\hat{s}_X$. There exist a dual correlation parameter, $\tilde{\rho}_{n,m}$ and a dual optimal Kelly strategy $\tilde{w} = (\tilde{w}_n, \tilde{w}_m)'$ that generate the same magnitude of the optimal instantaneous Sharpe ratio of the portfolio. The dual parameters are given by*

$$\tilde{\rho}_{n,m} (t) = \frac{1 - \rho_{n,m} (t) \Phi (t)}{\Phi (t) - \rho_{n,m} (t)}, \quad \Phi (t) = \frac{1}{2} \left(\rho_{n,m}^* (t) + \frac{1}{\rho_{n,m}^* (t)} \right),$$

while the dual optimal Kelly strategy $\tilde{w}$ equals

$$\begin{pmatrix} \tilde{w}_n (t) \\ \tilde{w}_m (t) \end{pmatrix} = \begin{pmatrix} \hat{w}_n (t) \\ -\hat{w}_m (t) \end{pmatrix} \sqrt{\frac{1 - \rho_{n,m}^2 (t)}{1 - \tilde{\rho}_{n,m}^2 (t)}} \operatorname{sgn} \left(s_n^2 (t) - s_m^2 (t) \right)$$

Proof. We need to prove that

$$\frac{s_n^2 (t) - 2 s_n (t) \tilde{\rho}_{n,m} (t) s_m (t) + s_m^2 (t)}{1 - \tilde{\rho}_{n,m}^2 (t)} = \frac{s_n^2 (t) - 2 s_n (t) \rho_{n,m} (t) s_m (t) + s_m^2 (t)}{1 - \rho_{n,m}^2 (t)} = \hat{s}_X^2 (t) .$$

Straightforward but tedious calculations show that the dual correlation can be obtained by a reflection in the first order condition for the equilibrium correlation.

$$\tilde{\rho}_{n,m} (t) = 2 \frac{s_n (t) s_m (t)}{\hat{s}_X^2 (t)} - \rho_{n,m} (t) .$$

It further follows from Eq. (50) that the portfolio corresponding to the dual correlation satisfies

$$\frac{\tilde{w}_n^2 (t)}{\hat{w}_n^2 (t)} = \frac{\tilde{w}_m^2 (t)}{\hat{w}_m^2 (t)} = \frac{1 - \rho_{n,m}^2 (t)}{1 - \tilde{\rho}_{n,m}^2 (t)}.$$

The proof concludes from a close inspection of the two possible cases for the equilibrium correlation ρ^*. □

6.3 Several risky assets and bank account game

By pasting together the individual pieces we can construct a full equilibrium covariance matrix and an associated dual covariance matrix. Similar to the definition $\sigma_{diag} = \operatorname{diag} (\sigma)$ we introduce the diagonal volatility matrices

$$\sigma_{diag}^* = \operatorname{diag} (\sigma^*) , \quad \tilde{\sigma}_{diag} = \operatorname{diag} (\tilde{\sigma}) , \tag{60}$$

where $\sigma^* = (\sigma_1^*, \ldots, \sigma_N^*)'$ contains the equilibrium volatilies and $\tilde{\sigma} = (\tilde{\sigma}_1, \ldots, \tilde{\sigma}_N)'$ the associated dual volatilities. We now define the corresponding asset-asset covariance matrices according to

$$V^* (t) = \sigma_{diag}^* (t) \rho^* (t) \sigma_{diag}^* (t) , \quad \tilde{V} (t) = \tilde{\sigma}_{diag} (t) \tilde{\rho} (t) \tilde{\sigma}_{diag} (t) , \tag{61}$$

in terms of the equilibrium and dual correlation matrices. We let $s^* = (s_1^*, \ldots, s_N^*)'$ and $\tilde{s} = (\tilde{s}_1, \ldots, \tilde{s}_N)'$ denote the equilibrium and the dual instantaneous Sharpe ratios, respectively. The components then relates to the equilibrium volatility σ^* and to the instantaneous Sharpe ratio s according to

$$s_n^* (t) = \sigma_n^* (t) e_n (t) , \quad \tilde{s}_n (t) = s_n (t) (2 e_n (t) - 1) , \quad e_n (t) = \frac{1}{2} (1 + \operatorname{sgn} (\mu_n (t) - r (t))) . \tag{62}$$

This completes the specification of the equilibrium and dual market.

Now, let us consider an optimal Kelly trader who can invest in the bank account and all the risky assets. As demonstrated in Proposition 3.2 and Theorem 6.3 the logarithmic drift then satisfies

$$\max_{w(t)} \mu_X(t) = r(t) + \frac{1}{2}\hat{s}_X^2(t) \geq r(t) + \frac{1}{2}\max_{1\leq n\leq N} s_n^2(t) = \min_{\rho(t)}\max_{w(t)} \mu_X(t), \tag{63}$$

such that, in equilibrium, the market coalition sets the correlation in accordance with the lower bound and thereby reducing the outcome to a one risky asset and bank account game. We summarize the result below.

Proposition 6.5. *Given an optimal Kelly trader with local portfolio dynamics*

$$\mu_X(t) = r(t) + \frac{1}{2}\hat{s}_X^2(t), \quad \sigma_X^2(t) = \hat{s}_X^2(t).$$

If the correlation market is in equilibrium then

$$\hat{s}_X^2(t) = \max_{1\leq n\leq N} s_n^2(t),$$

while if, in addition, the volatility market is in equilibrium then

$$\hat{s}_X^2(t) = 2\left(\max_{1\leq n\leq N} \mu_n(t) - r(t)\right)_+.$$

Proof. The results follow from Theorem 3.1, Eq. (42) and Eq. (63). □

When the correlation market is in equilibrium an investor can do no better than to trade in the risky asset with the highest Sharpe ratio magnitude. In this case there is no extra gain to be made from diversification. What should be mentioned though is that if several assets share the same instantaneous Sharpe ratio the equilibrium point is degenerated with the correlation between those assets being equal to one. It therefore follows that any weighted average of such assets (assuming the weights sum to one) yields the same outcome as simply investing in one of them. In practise, however, when the instantaneous Sharpe ratio must be estimated from historical data it is not obvious how to single out the risky assets with the highest instantaneous Sharpe ratio magnitudes. As a consequence an investor might want to be prudent and take positions in more risky assets than what is otherwise theoretically optimal.

In Table 1 we present sample estimates for the total return indices (dividends re-invested) of S&MP500 and Nasdaq Composite using daily data for the last 15 years. The result shows that the equilibrium correlation is of the right order while the equilibrium volatility is far higher than what was realized historically. This leads us to suspect that the market participants (on average) are not following an optimal Kelly strategy as such a trading strategy is deemed too risky.

Asset	μ	σ	ρ	s	σ^*	ρ^*
S&P500	8.52%	18.46%	95.47%	0.4725	37.47%	87.49%
Nasdaq Composite	10.36%	20.18%	95.47%	0.3736	42.10%	87.49%

Table 1: The parameters (μ, σ, ρ) are estimated over the time interval 2005Jan-2019Dec (source: Bloomberg). We also estimate the average interest rate $r = 1.50\%$ over this interval. The remaining parameters are calculated based on the estimates.

In the next section we therefore investigate the dynamic stability of the equilibrium points from the perspective of a Kelly trader employing an arbitrary Kelly multiplier.

7 Dynamic Stability in the Covariance Market

In this section we show that the equilibrium points derived for the covariance market are stable in the sense that an optimal Kelly trader, and most Kelly traders, will force the volatility level towards equilibrium., while all Kelly traders will force the correlation level towards equilibrium. In order to understand the mechanism one has to consider the order book of a particular asset. For a given point in time the order book contains one sequence of bid prices with associated quantities and one sequence of offer prices with associated quantities. A trader who buys a larger quantity of an asset than what is provided for at the lowest offer price will move the price of the asset upwards. Similarly, a trader who sells a larger quantity of an asset than what is asked for at the highest bid price will move the price of the asset downwards. It is this joint interaction between the market participants that causes the price level to move around in a seemingly random way. The more liquid the market, that is the more participants, the more accurately can the randomness governing the price evolution be described by a Brownian motion as in Eq. (6).

In order to launch the stability analysis we start by imposing some structure on the dynamics of the risky assets. We let $M = (M_1, \ldots, M_N)'$ represent a local martingale with components

$$M_n(t) = \int_0^t \Sigma'_{h(n)}(s)\, dW(s), \quad 1 \leq n \leq N, \tag{64}$$

for some permutation function h such that $h(1) = n$ and $h(2) = m$. Although the permutation rearranges the elements of the instantaneous covariance matrix, the new matrix $V_h = \Sigma_h \Sigma'_h$ is still a.s. positive definite and therefore admits a Cholesky decomposition of the form $V_h = LL'$, where L is a unique lower triangular matrix. We now let $Z = (Z_1, \ldots, Z_M)'$ denote a standard Brownian motion with the first N components given by

$$Z(t) = \int_0^t L^{-1}(s)\, \Sigma'_h(s)\, dW(s), \tag{65}$$

such that $dM = \Sigma_h dW = LdZ$. Furthermore, the remaining components of Z can always be constructed such that any $\mathbb{F}^W$-adapted process is also $\mathbb{F}^Z$-adapted. The purpose of this manipulation is to separate the Brownian components driving the randomness of the risky assets from those that makes the parameters of the model measurable. This enables us to write

$$\frac{dP_n(t)}{P_n(t)} = b_n(t)\, dt + \sigma_n(t)\, dZ_1(t), \tag{66}$$

$$\frac{dP_m(t)}{P_m(t)} = b_m(t)\, dt + \sigma_m(t) \left(\rho_{n,m}(t)\, dZ_1(t) + \rho^{\perp}_{n,m}(t)\, dZ_2(t)\right), \quad \rho^{\perp}_{n,m}(t) = \sqrt{1 - \rho^2_{n,m}(t)}. \tag{67}$$

If we want to study the markets response to shocks related to the Brownian motion Z then, as shown in [1], the right tool to use for this application is Malliavin calculus. We intentionally keep the technical details to a minimum and simply state that for any $\mathbb{F}$-measurable random variable Y the Malliavin derivative $D_l(s)\, Y$ describes the response of Y to a shock in Z_l at time s. Hence, for a fixed time s, the Malliavin derivative $D(s) = (D_1(s), \ldots, D_M(s))'$ is a vector operator, normalized such that $D_l(s)\, Z_l(t) = \mathbf{1}\{s \leq t\}$, $1 \leq l \leq M$. Each component of the Malliavin derivative further satisfies the classical chain rule, that is $D_l(s)\, F(Y) = F'(Y)\, D_l(s)\, Y$, provided the derivative of F is well defined. With this information at hands we now proceed.

We let q_n denote the number of units held in the asset P_n, while q_m denotes the number of units held in the asset P_m. The wealth fractions are similarly defined by $q_n P_n = w_n X$ and $q_m P_m = w_m X$ such that the dynamics of the portfolio process can be expressed as

$$\frac{dX(t)}{X(t)} = (1 - w_n(t) - w_m(t))\, r(t)\, dt + w_n(t) \frac{dP_n(t)}{P_n(t)} + w_m(t) \frac{dP_m(t)}{P_m(t)}. \tag{68}$$

We also assume that the wealth fractions (w_n, w_m) are locally uncorrelated with the asset prices (P_n, P_m). This means that the quadratic covariance process $[w_n, P_n] = 0$ and similar for the other combinations. We refer to [7] for a precise definition of the quadratic covariation process and claim that in our setup, where all stochastic processes are continuous, the term $d[w_n, P_n]$ can be identified with the Itô product $dw_n dP_n$. We further stress that the assumption made can be relaxed to accommodate particular model specifications if needed. However, in order to simplify the analysis, we have decided to omit such extensions in this analysis. The market response to a shock in the Brownian motion Z can now be evaluated and since we are only concerned about instantaneous responses the Malliavin derivatives are particularly easy to compute.

Lemma 7.1. *Suppose that the quadratic covariation process between the trading strategies and the asset prices vanishes, that is $[w_i, P_j] = 0$, for $i, j \in \{n, m\}$. The response of the asset holdings (q_n, q_m) to a shock of the Brownian motion Z then equals*

$$D_1(t)\, q_n(t) = \frac{X(t)}{P_n(t)} \left(\sigma_n(t)\, w_n(t)\, (w_n(t) - 1) + \rho_{n,m}(t)\, \sigma_m(t)\, w_n(t)\, w_m(t)\right),$$

$$D_2(t)\, q_n(t) = \frac{X(t)}{P_n(t)} \rho^{\perp}_{n,m}(t)\, \sigma_m(t)\, w_n(t)\, w_m(t),$$

$$D_1(t)\, q_m(t) = \frac{X(t)}{P_m(t)} \left(\rho_{n,m}(t)\, \sigma_m(t)\, w_m(t)\, (w_m(t) - 1) + \sigma_n(t)\, w_n(t)\, w_m(t)\right),$$

$$D_2(t)\, q_m(t) = \frac{X(t)}{P_m(t)} \rho^{\perp}_{n,m}(t)\, \sigma_m(t)\, w_m(t)\, (w_m(t) - 1).$$

Proof. We first observe that the instantaneous impulse response of the risky assets are given by

$$D_1(t)\,P_n(t) = \sigma_n(t)\,P_n(t)\,, \quad D_2(t)\,P_n(t) = 0,$$

$$D_1(t)\,P_m(t) = \rho_{n,m}(t)\,\sigma_m(t)\,P_m(t)\,, \quad D_2(t)\,P_m(t) = \rho^{\perp}_{n,m}(t)\,\sigma_m(t)\,P_m(t)\,,$$

which implies that the impulse response of the portfolio process takes the form

$$D_1(t)\,X(t) = X(t)\,(w_n(t)\,\sigma_n(t) + w_m(t)\,\rho_{n,m}(t)\,\sigma_m(t))\,,$$

$$D_2(t)\,X(t) = X(t)\,w_m(t)\,\rho^{\perp}_{n,m}(t)\,\sigma_m(t)\,.$$

We now illustrate the proof for the first result. It follows from the chain rule that

$$D_1(t)\,q_n(t) = \frac{X(t)}{P_n(t)}D_1(t)\,w_n(t) + \frac{w_n(t)}{P_n(t)}D_1(t)\,X(t) - \frac{w_n(t)\,X(t)}{P_n^2(t)}D_1(t)\,P_n(t)\,,$$

where the term $D_1 w_n$ vanishes due to the assumption that $[w_n, P_n] = 0$. The other results are proved in a similar way. □

In order to distinguish between various trader types we say that a trader is a fractional trader if $0 < w < 1$ and a leverage trader if either $w < 0$ or $w > 1$. In general, it is of course possible for a trader to be fractional in one asset and leveraged in another. We further say that a pair-trader is long-short if $w_n w_m < 0$ and long/short-only if $w_n w_m > 0$. In the particular case when the risky assets are locally uncorrelated, that is $\rho_{n,m} = 0$, we see that the asset holdings response to a shock is negative for fractional and long-short traders, while positive for leveraged and long/short-only traders.

We now follow a path different from what is taken elsewhere in this paper, that is we assume that an investor's activity has an impact on the asset prices in the sense that buying an asset pushes the price up while selling an asset pushes the price down. The implication of such a feedback rule is best understood by a simple example. Suppose that some trader executes a trade. The price level will then jump either up or down depending on whether he posted a buy or a sell order. We can regard such an event as a shock to the Brownian motion Z. This implies that our trader no longer has the fraction w of his wealth invested in the risky asset and therefore needs to buy or sell additional units q, which in turn affects the price level. If we further assume that the adjustment occurs momentously in time this generates a feedback mechanism that we take to be of the form

$$D_l(t)\,P_i(t) \to D_l(t)\,P_i(t) + \gamma_i(t)\,\frac{P_i^2(t)}{X(t)}D_l(t)\,q_i(t)\,, \quad \gamma_i(t) \geq 0, \quad i \in \{n, m\}, \quad l \in \{1, 2\}. \tag{69}$$

In order to relate the feedback mechanism to the model parameters it is often more convenient to work with the quadratic covariation process. The two concepts connect as described below.

$$\frac{d}{dt}[P_i, P_j](t) = D_1(t)\,P_i(t)\,D_1(t)\,P_j(t) + D_2(t)\,P_i(t)\,D_2(t)\,P_j(t)\,, \quad i, j \in \{n, m\}. \tag{70}$$

Moreover, it is reasonable to assume that the feedback mechanism is small and by this we mean that the components of $\gamma = (\gamma_n, \gamma_m)'$ are close to zero. If we ignore all terms of order $\mathcal{O}\left(\gamma^2\right)$ it follows from Lemma 7.1 and Eq. (70) that

$$\sigma_n^2(t) \to \sigma_n^2(t) + 2\gamma_n(t)\left(\sigma_n^2(t)\,w_n(t)\,(w_n(t) - 1) + \rho_{n,m}(t)\,\sigma_n(t)\,\sigma_m(t)\,w_n(t)\,w_m(t)\right), \tag{71}$$

$$\sigma_m^2(t) \to \sigma_m^2(t) + 2\gamma_m(t)\left(\sigma_m^2(t)\,w_m(t)\,(w_m(t) - 1) + \rho_{n,m}(t)\,\sigma_n(t)\,\sigma_m(t)\,w_n(t)\,w_m(t)\right). \tag{72}$$

It is of particular interest to consider an investor who only trades in one risky asset and the bank account. Such an investor, described by $w_n w_m = 0$, can be seen to dampen the market volatility if his position is fractional and amplify the market volatility if his position is leveraged

$$\sigma_i^2(t) \to \sigma_i^2(t) + 2\gamma_i(t)\,\sigma_i^2(t)\,w_i(t)\,(w_i(t) - 1)\,, \quad i \in \{n, m\}. \tag{73}$$

In terms of the order book this means that a fractional trader, who sells on positive price jumps and buys on negative price jumps, can naturally place his orders beforehand. A leveraged trader, however, ends up chasing the market and can only post his orders after the price jumps. We now turn to analyze how the feedback mechanism impacts the correlation. As shown in the Appendix the update is given by

$$\rho_{n,m}(t) \to \rho_{n,m}(t) + \left(1 - \rho_{n,m}^2(t)\right)\left(\gamma_n(t)\,\frac{\sigma_m(t)}{\sigma_n(t)} + \gamma_m(t)\,\frac{\sigma_n(t)}{\sigma_m(t)}\right)w_n(t)\,w_m(t) + \mathcal{O}\left(\gamma^2(t)\right). \tag{74}$$

The interpretation is that a long-short trader decreases the market correlation while a long/short-only trader increases the market correlation. The overall impact of the feedback mechanism depends, of course, on the aggregate positions of all the market participants. Hence, little can be said, in general, of such a diverse group that might include both partly and fully irrational investors. Instead, we focus on the homogeneous group of Kelly traders. Below we show that not only will this group force the market towards equilibrium, they will also increase their relative proportion of the outstanding asset shares.

7.1 One risky asset and bank account game

In this section we provide the specific details of the convergence towards the equilibrium volatility. Underlying the argument is the observation that fractional traders decrease volatility while leveraged traders increase volatility.

Proposition 7.2. *Consider an efficient Kelly trader who only invests in the bank account and one risky asset P_n. The trading strategy $w_n = k\hat{w}_n$, $k \in (0,1]$, then satisfies*

$$\begin{array}{rcc} 0 < w_n(t) < 1; & \sigma_n(t) > \sigma_n^*(t), & \\ 0 < w_n(t) < 1; & \Psi(t)\sigma_n^*(t) < \sigma_n(t) < \sigma_n^*(t), & \mu_n(t) > r(t), \\ w_n(t) > 1; & \sigma_n(t) < \Psi(t)\sigma_n^*(t) < \sigma_n^*(t), & \mu_n(t) > r(t), \\ w_n(t) < 0; & \sigma_n(t) < \sigma_n^*(t), & \mu_n(t) < r(t), \end{array}$$

where $\Psi = \sqrt{k/(2-k)}$.

Proof. We express the optimal Kelly strategy $\hat{w}_n = s_n/\sigma_n$ according to

$$\hat{w}_n(t) = \frac{1}{2} + \frac{\mu_n(t) - r(t)}{\sigma_n^2(t)} = \frac{1}{2} + \frac{2e_n(t) - 1}{2}\left(\frac{\sigma_n^*(t)}{\sigma_n(t)}\right)^2, \quad e_n(t) = \frac{1}{2}\left(1 + \operatorname{sgn}\left(\mu_n(t) - r(t)\right)\right).$$

By analyzing each branch separately we then find that

$$\sigma_n(t) > \sigma_n^*(t) \Rightarrow \begin{cases} 0 < \hat{w}_n(t) < \frac{1}{2}, & \mu_n(t) < r(t) \\ \frac{1}{2} < \hat{w}_n(t) < 1, & \mu_n(t) > r(t) \end{cases}, \quad \sigma_n(t) < \sigma_n^*(t) \Rightarrow \begin{cases} \hat{w}_n(t) < 0, & \mu_n(t) < r(t) \\ \hat{w}_n(t) > 1, & \mu_n(t) > r(t) \end{cases}.$$

In the particular case when $\mu_n = r$ we see that $\sigma_n > \sigma_n^*$, since then $\sigma_n^* = 0$, with $\hat{w}_n = 0.5$. The proof concludes by transforming the inequalities for the optimal Kelly strategy to those of a Kelly trader. □

We see that if the market volatility is greater than the equilibrium volatility any efficient Kelly trader will employ a fractional strategy and consequently force the market volatility down towards the equilibrium point. However, if the market volatility is less than the equilibrium volatility the situation is more complex. If $\mu_n < r$ any efficient Kelly trader will leverage and force the market volatility up towards the equilibrium. If, on the other hand, $\mu_n > r$ only the efficient Kelly traders with sufficiently high Kelly multiplier will force the market volatility up towards the equilibrium volatility. Those Kelly traders employing a low Kelly multiplier will instead further decrease the market volatility away from the equilibrium point. This shows that in a highly liquid market, consisting of efficient Kelly traders only, the market volatility cannot be greater than the equilibrium volatility. However, the market volatility can be less than the equilibrium volatility if the aggregate risk appetite is not sufficiently high.

Corollary 7.3. *The equilibrium volatility for a Kelly trader, with Kelly multiplier k, is given by*

$$\sigma_n^*(t) = \sqrt{2\left(r(t) - \mu_n(t)\right)_+} + \Psi(k(t))\sqrt{2\left(\mu_n(t) - r(t)\right)_+}, \quad \Psi(k) = \sqrt{k/(2-k)},$$

which implies that, in equilibrium, the instantaneous Sharpe ratio equals

$$s_n^*(t) = \frac{1}{k(t)}\sigma_n^*(t)\, e_n(t), \quad e_n(t) = \frac{1}{2}\left(1 + \operatorname{sgn}\left(\mu_n(t) - r(t)\right)\right)$$

Proof. The proof follows from Proposition 7.2 and straightforward calculations. □

In Table 2 we compute the implied Kelly multipliers corresponding to representative Kelly traders of the indices S&P500 and Nasdaq Composite. This shows that the two indices are traded very consistently; there is just as much appetite to leverage either of the indices.

Asset	μ	σ	s	k	$\bar{k}$	σ^*
S&P500	8.52%	18.46%	0.4725	0.3907	0.3844	18.21%
Nasdaq Composite	10.36%	20.18%	0.5401	0.3736	0.3844	20.46%

Table 2: The parameters (μ, σ, s) are similar to those in Table 1. We also use the average interest rate $r = 1.50\%$ such that the implied Kelly multiplier takes the form $k = \sigma/s$. The equilibrium volatility is based on the average Kelly multiplier $\bar{k}$.

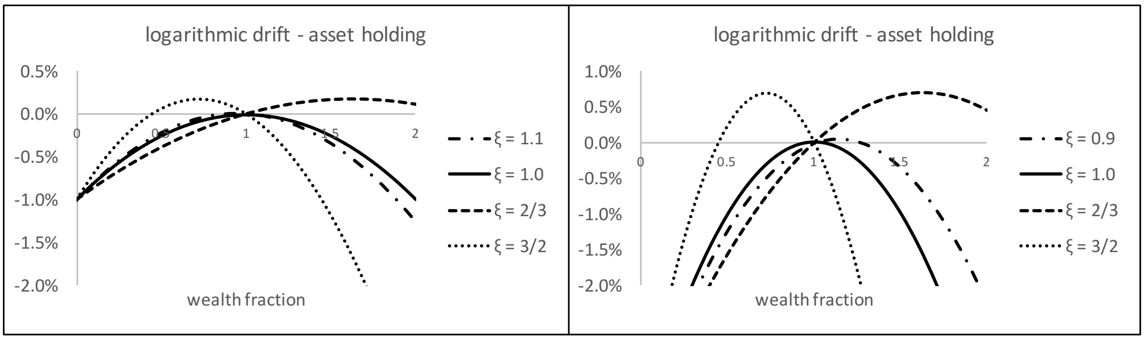

Figure 5: The plots show the logarithmic drift of the asset holdings q_i, $i \in \{n, m\}$, as a function of a constant wealth fraction w_i for various levels of volatility σ_i. The market parameters are defined as in Fig. 1 and used to compute the equilibrium volatility σ_i^*. The additional volatility levels are derived from the expression $\sigma_i = \xi \sigma_i^*$.

From a practical point of view the knowledge of the equilibrium volatility is of utmost importance since it determines whether or not a trader should take on a fractional or leveraged position. We illustrate the outcome in Fig. 5 by plotting the drift of $\log(q_i)$, which assuming the wealth fraction w_i to be constant equals the drift of $\log(X/P_i)$, $i \in \{n, m\}$. By knowing whether or not it is optimal to be fractional or leveraged a trader will on average increase his total holdings of the assets. The maximal increase is obtained by employing the optimal Kelly strategy, however, a trader will still increase his asset holdings given that he does not deviate too far from the optimal strategy. Only when the market volatility is in equilibrium must the optimal Kelly strategy be used if the asset holdings are not to diminish. Fig. 5 also highlights the role of the dual volatility. If the market volatility is of the form $\sigma_i = \xi_i \sigma_i^*$ the dual volatility equals $\tilde{\sigma}_i = \xi_i^{-1} \sigma_i^*$. Hence, we can interpret the dual volatility as the volatility at which there is always an opposite trading strategy that does equally well in terms of accumulating asset shares. This shows that as time goes by the group of Kelly traders will acquire more and more assets, which reinforces the aggregate feedback mechanism and thus the speed towards the equilibrium point.

7.2 Two risky assets and bank account game

In this section we provide the specific details of the convergence towards the equilibrium correlation. Underlying the argument is the observation that long-short traders decrease the correlation while long/short-only traders increase the correlation.

Proposition 7.4. *Consider an efficient Kelly trader who only invests in the bank account and two risky assets, say P_n and P_m. The trading strategy $w = k\hat{w}$, $k \in (0, 1]$, then satisfies*

$$\begin{aligned} w_n(t)\, w_m(t) > 0; &\quad \rho_{n,m}(t) < \rho^*_{n,m}(t), \\ w_n(t)\, w_m(t) < 0; &\quad \rho_{n,m}(t) > \rho^*_{n,m}(t). \end{aligned}$$

Proof. According to Eq. (50) the optimal Kelly strategy takes the form

$$\hat{w} = (\hat{w}_n, \hat{w}_m)' = \frac{1}{1 - \rho^2_{n,m}(t)} \left(\frac{s_n(t) - \rho_{n,m}(t)\, s_m(t)}{\sigma_n(t)}, \frac{s_m(t) - \rho_{n,m}(t)\, s_n(t)}{\sigma_m(t)} \right)'.$$

In terms of the equilibrium correlation $\rho^*_{n,m}$, defined in Eq. (54), one easily shows that

$$\hat{w}_n(t)\, \hat{w}_m(t) = \frac{1}{\left(1 - \rho^2_{n,m}(t)\right)^2} \frac{\max\left(s_n^2(t), s_m^2(t)\right)}{\sigma_n(t)\, \sigma_m(t)} \left(\rho^*_{n,m}(t) - \rho_{n,m}(t)\right) \left(1 - \rho^*_{n,m}(t)\, \rho_{n,m}(t)\right).$$

Hence, the sign of $\hat{w}_n \hat{w}_m$ is determined by the sign of $\rho^*_{n,m} - \rho_{n,m}$. The proof concludes by transforming the result to a Kelly trader. □

We see that if the market correlation is less than the equilibrium correlation any Kelly trader is a long/short-only trader and will consequently force the market correlation up toward the equilibrium point. Similarly, we see that if the market correlation is greater than the equilibrium correlation any Kelly trader is a long-short trader. The group of Kelly traders will in this case force the market correlation down towards the equilibrium point. Hence, different to the equilibrium volatility the equilibrium correlation is not affected by the Kelly multiplier.

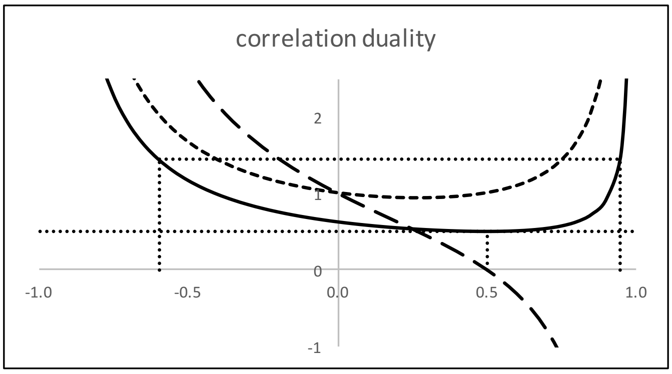

Figure 6: The plot shows the excess logarithmic drift $\mu_X - r$ of an optimal Kelly trader (solid line) as a function of the correlation parameter $\rho_{n,m}$. The parameters used are $\sigma_n = s_n = 1.0$ and $\sigma_m = s_m = 0.5$ such that $\rho^*_{n,m} = 0.5$. We also plot the optimal trading strategies $\hat{w}_n$ (dotted line) and $\hat{w}_m$ (dashed line).

In Fig. 6 we plot the excess rate of logarithmic return $\mu_X - r = \frac{1}{2}\hat{s}_X^2$ for an optimal Kelly trader together with the optimal Kelly strategies $\hat{w}_n$ and $\hat{w}_m$. As demonstrated in Corollary 6.4, we see that for each correlation $\rho_{n,m} \lesseqgtr \rho^*_{n,m}$ there exists a dual correlation $\tilde{\rho}_{n,m} \gtreqless \rho^*_{n,m}$ that maintains the magnitude of the optimal Sharpe ratio. If $\rho_{n,m}$ generates a long/short-only strategy then $\tilde{\rho}_{n,m}$ corresponds to a long-short strategy and vice versa. Only when the correlation is in equilibrium will a Kelly trader choose to trade exclusively in one of the assets, that is in the asset with the highest Sharpe ratio magnitude.

8 Equilibrium in the Rates Market

In this section we look at the role played by a central bank in charge of setting the interest rate r. We start by considering the interaction between the central bank and optimal Kelly traders. Thereafter, we investigate the central bank's response to a collection of Kelly traders. In order to get started let us first consider the situation when the full covariance market is in equilibrium with respect to an optimal Kelly trader. This implies that an optimal Kelly trader will locally be fully invested in either the bank account or in the particular asset with the highest squared Sharpe ratio. The decision is triggered by whether $\max_n \mu_n$ is less or greater than the interest rate r as described in Proposition 6.5 and summarized below.

Lemma 8.1. *Suppose that the full covariance matrix is in equilibrium with respect to an optimal Kelly trader. Then, for any interest rate r, we have*

$$\begin{aligned} r^*(t) &= r(t); && \max_n \mu_n(t) > r(t), \\ r^*(t) + \sigma^2_{\min}(t) &= r(t); && \max_n \mu_n(t) \leq r(t). \end{aligned}$$

Proof. If $\max_n \mu_n > r$ the equilibrium trading strategy is an assets-only strategy. In order for this strategy to also be Kelly optimal it follows from Corollary 5.3 that $\sigma^2_{\min} = b_{\min} - r$. If $\max_n \mu_n \leq r$ the optimal instantaneous Sharpe ratio $s_X = 0$. It now follows from Proposition 5.2 that $\sigma_0 = 0$ and $b_{\min} = r$. The proof concludes by expressing $b_{\min}$ in terms of r^*. □

Hence, if all investors are optimal Kelly traders and the full covariance market is in equilibrium the rate r^* will, in general, automatically adjust to the level of the interest rate r set by the central bank.

Let us now consider the situation where the full covariance market is not in equilibrium. In this case the covariance matrix V is exogenously given rather than being related to the logarithmic drift μ and the interest rate r. What makes the search for an equilibrium interest rate different to the previous analysis for the assets is that investors only have access to the interest rate r via the locally risk-free bank account. The absence of volatility implies that there is no market mechanism that will drive the interest rate towards an equilibrium point. However, such a point do exist as shown below.

Theorem 8.2. *Given an optimal Kelly trader who invests in the bank account and the risky assets. Then*

$$\min_{r(t)} \max_{w(t)} \mu_X(t) = \max_{w(t)} \min_{r(t)} \mu_X(t) = r^*(t) + \frac{1}{2}\left(\sigma_0^2(t) + \sigma^2_{\min}(t)\right).$$

The equilibrium solution is given by $(r^*, \hat{w})$, while the corresponding volatility equals

$$\sigma_X^2(t) = \sigma_0^2(t) + \sigma_{\min}^2(t).$$

Proof. It follows from Theorem 5.2 that the logarithmic drift and the volatility of an optimal Kelly strategy, expressed in terms of r^* instead of $b_{\min}$, take the form

$$\mu_X(t) = r^*(t) + \frac{1}{2}\left(\sigma_0^2(t) + \sigma_{\min}^2(t)\right) + \frac{1}{2}\left(\frac{r(t) - r^*(t)}{\sigma_{\min}(t)}\right)^2,$$

$$\sigma_X^2(t) = \sigma_0^2(t) + \sigma_{\min}^2(t) + \left(\frac{r(t) - r^*(t)}{\sigma_{\min}(t)}\right)^2 - 2\left(r(t) - r^*(t)\right).$$

Hence, we are left to prove that $\max_w \min_r \mu_X = \min_r \max_w \mu_X$. We now consider Eq. (16) for an arbitrary trading strategy

$$\mu_X(t) = r(t) + w'(t)\left(b(t) - r(t)\mathbf{1}_N\right) - \frac{1}{2}w'(t)V(t)w(t).$$

From the first order condition, with respect to r, it then follows that $\mathbf{1}_N' w = 1$, which implies that an assets-only portfolio must be used. By the use of Corollary 5.3 it now follows that the trading strategy is of the form $v_1 u^1 + u^2$. The proof concludes from the observation that such a strategy is Kelly optimal if $v_1 = 1$ and that this choice implies the interest rate r^*. □

It follows that, in a market consisting of optimal Kelly traders only, there exists an equilibrium interest rate r^* such that if the central banks sets $r = r^*$ then all available capital will be invested in the risky assets. Hence, in this case, the central bank can influence the asset allocation in such a way as to mimic the outcome corresponding to the covariance market being in equilibrium.

Of course, the market does not consist of optimal Kelly traders only. For this reason we take a more global approach to analyzing the central bank's decision problem. Henceforth, we assume that there are J investors, each being a Kelly trader, and we let $X = (X_1, \ldots, X_J)'$ represent the portfolio wealth of the investors. The investors can trade in the capital market $(B, P_1, \ldots, P_N)$ and we assume they agree upon the model parameters being used. We let

$$X_d(t) = \sum_{j=1}^{J} X_j(t) = \mathbf{1}_J' X(t), \tag{75}$$

denote the aggregate demand for investments and notice that the dynamic evolution can be expressed as

$$\frac{dX_d(t)}{X_d(t)} = \sum_{j=1}^{J} \frac{X_j(t)}{X_d(t)} \frac{dX_j(t)}{X_j(t)}. \tag{76}$$

Since each investor is a Kelly trader we know from Theorem 5.2 and Eq. (37) that his portfolio can be decomposed into a dynamic trading position in the mutual fund corresponding to the strategy u^3 and the bank account. In other words, for each Kelly trader $j \in \{1, \ldots, J\}$, characterized by a Kelly multiplier k_j, we have

$$\frac{dX_j(t)}{X_j(t)} = v_j(t)\frac{dX_{u_3}(t)}{X_{u_3}(t)} + \left(1 - v_j(t)\right)\frac{dB(t)}{B(t)}, \quad v_j(t) = k_j(t)\left(1 - \frac{r(t) - r^*(t)}{\sigma_{\min}^2(t)}\right). \tag{77}$$

This shows that the aggregate demand is of the same form

$$\frac{dX_d(t)}{X_d(t)} = v_d(t)\frac{dX_{u_3}(t)}{X_{u_3}(t)} + \left(1 - v_d(t)\right)\frac{dB(t)}{B(t)}, \quad v_d(t) = \frac{1}{X_d(t)}\sum_{j=1}^{J} v_j(t) X_j(t). \tag{78}$$

Hence, we may think of the aggregate demand as the portfolio corresponding to a representative Kelly trader. If we set $k = (k_1, \ldots, k_J)'$ the aggregate trading strategy then relates to the optimal Kelly strategy as shown below

$$w_d(t) = k_d(t)\hat{w}(t), \quad k_d(t) = \frac{k'(t)X(t)}{\mathbf{1}_J' X(t)}. \tag{79}$$

It follows that the central bank can steer away investments from the bank deposits by setting the interest rate to

$$r(t) = r^*(t) - \frac{1 - k_d(t)}{k_d(t)}\sigma_{\min}^2(t), \tag{80}$$

since such a choice implies that $\mathbf{1}'_N w_d = 1$. Whether the goal of the central bank should be to set the interest rate in such a way is of course debatable. There are nevertheless valid reasons for why the government would want to discourage investors to invest in the bank account. If we assume that most of a country's GDP-growth can be explained by either domestic consumption or the success of private firms, one can argue that it is in the government's best interest to channel capital to where it contributes the most. If we further assume that the market representative Kelly multiplier $k_d \in [0, 1]$ we see that the interest rate set by the central bank cannot exceed r^*. Moreover, the less appetite the major investors have to leverage their positions, that is the lower the representative Kelly multiplier, the higher the odds that the central bank will enter into a negative rate territory.

9 Concluding Remarks

In this paper we show that the Kelly framework is the natural multi-period extension of the one-period mean-variance model of Markowitz. A Kelly trader selects his portfolio allocation along an instantaneous efficient frontier for which the instantaneous Sharpe ratio is maximal. The allocation can further be decomposed into the bank account and a particular assets-only mutual fund with the property that it has maximal instantaneous Sharpe ratio among all assets-only mutual funds. However, in contrast to the mean-variance approach the Kelly framework answers the fundamental question of leverage: it is never efficient to leverage more than the optimal Kelly strategy and if the leverage is twice that of the optimal Kelly strategy bankruptcy is likely to occur. One cannot stress enough the practical importance of such a statement and as a sideline we mention, without going into details, the notorious hedge fund Long-Term Capital Management which, allegedly at times, ran a debt-to-equity ratio of over 25 to 1.

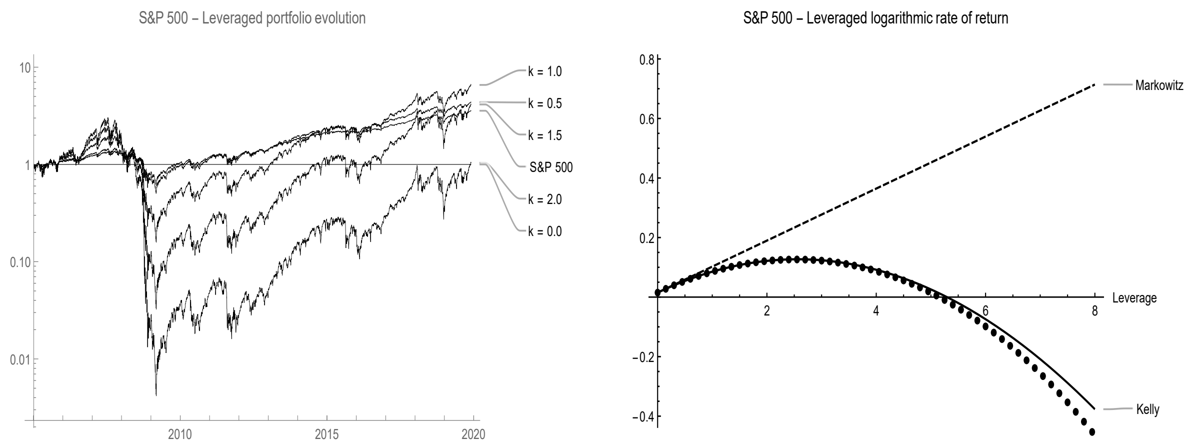

Figure 7: The first plot shows the total return index of S&P500 over the time interval 2005Jan-2019Dec using daily observations. The estimated logarithmic drift $\mu_1 = 8.52\%$ and the volatility $\sigma_1 = 18.46\%$, as in Table 1, while the interest rate $r = 1.50\%$. From the index we construct various Kelly portfolios by varying the Kelly multiplier. The second plot shows the realized logarithmic rate of return $\frac{1}{T}\log\left(X\left(T\right)/X\left(0\right)\right)$, dotted line, as a function of the leverage. For comparison we also plot the Kelly estimate $\frac{1}{T}\mathbb{E}\left[\log\left(X\left(T\right)/X\left(0\right)\right)\right]$ and the Markowitz estimate $\frac{1}{T}\log\left(\mathbb{E}\left[X\left(T\right)/X\left(0\right)\right]\right)$. Note that the optimal Kelly strategy, $k = 1$, corresponds to a leverage of $\hat{w}_1 = 2.56$.

The impact of leverage is illustrated in Fig. 7, where we plot the realized logarithmic rate of return for Kelly trading the S&P500 index. The first plot confirms the theoretical foundations of Theorem 3.1, namely that portfolios with Kelly multipliers of the form $k = 1 \pm c$ yield the same logarithmic rate of return but with different volatilities. The second plot shows the realized logarithmic return R_X^{cc} as a function of the leverage. It is interesting to note that the geometric mean $\exp\left(\mathbb{E}\left[\log\left(X\left(T\right)\right)\right]\right)$, which for deterministic coefficients $\left(\mu_X, \sigma_X\right)$ coincides with the median, predicts the portfolio outcome $X\left(T\right)$ far better than the mean $\mathbb{E}\left[X\left(T\right)\right]$ itself. This leads us to join the critics of modern portfolio theory who argue that the mean-variance model of Markowitz focuses on the wrong quantities. However, while the majority of those voices are unhappy with the variance representing portfolio risk, we primarily question the validity of imposing a target on the mean when the trading strategy essentially delivers the median. To be precise, our main criticism of the mean-variance model is that it lures investors into falsely believing that the portfolio will generate a profit in accordance with the targeted mean.

We further show that the expected utility approach of von Neumann-Morgenstern is embedded in the Kelly framework since we can always choose the Kelly multiplier to equal the reciprocal of the relative risk aversion coefficient for any strictly increasing and strictly concave utility function. Since any affine utility transformation generates the same Kelly multiplier, it follows that there is no natural utility scale that investors can refer to when communicating. Moreover, similar to the mean-variance model, the expected utility framework does not provide any details about how to leverage. The Kelly framework addresses the short-comings of the expected utility approach by first considering an objective rate of return measure and secondly employing a Kelly multiplier to control the leverage. Hence, the fact that the continuously compounded rate of return can be identified with a logarithmic utility function does not mean that the Kelly framework is a special case of the expected utility framework. It only tells us that the optimal Kelly strategy is also contained in the expected utility framework.

It is precisely the separation between return and leverage that allows us to study the interactions between various investors. By having an objective scale for the return we can formulate the market interactions as reduced two-person zero sum differential games. The solution to these games characterizes the market equilibrium. We show that the equilibrium points are dynamically stable, enforceable by Kelly traders, and that the knowledge of the equilibrium points determines what type of trading strategy an investor should employ. Consequently, market dynamics will force the covariance matrix to (or close to) an equilibrium that is fully specified by the logarithmic asset drift, the interest rate and the aggregate Kelly multiplier.

A Appendix

In this Appendix we provide details for how the feedback mechanism, introduced in section 7, influences the market correlation. We let $\bar{\sigma}_n$ and $\bar{\sigma}_m$ denote the volatilities of the risky assets P_n and P_m, when subject to the feedback mechanism. As stated section 7 these volatilities can be expressed as

$$\begin{aligned}\bar{\sigma}_n^2(t) &= \sigma_n^2(t) + 2\gamma_n(t)\left(\sigma_n^2(t)\,w_n(t)\,(w_n(t)-1) + \rho_{n,m}(t)\,\sigma_n(t)\,\sigma_m(t)\,w_n(t)\,w_m(t)\right),\\ \bar{\sigma}_m^2(t) &= \sigma_m^2(t) + 2\gamma_m(t)\left(\sigma_m^2(t)\,w_m(t)\,(w_m(t)-1) + \rho_{n,m}(t)\,\sigma_n(t)\,\sigma_m(t)\,w_n(t)\,w_m(t)\right),\end{aligned}$$

if we ignore all terms of order $\mathcal{O}\left(\gamma^2\right)$. The next result shows how the new volatilities relate to the old ones.

Lemma A.1. *For $0 \le \gamma_n, \gamma_m \ll 1$, the feedback mechanism influences the volatilities according to*

$$\begin{aligned}\frac{\sigma_n(t)}{\bar{\sigma}_n(t)}\frac{\sigma_m(t)}{\bar{\sigma}_m(t)} =& 1 - \gamma_n(t)\left(w_n(t)\,(w_n(t)-1) + \rho_{n,m}(t)\,\frac{\sigma_m(t)}{\sigma_n(t)}w_n(t)\,w_m(t)\right)\\ &- \gamma_m(t)\left(w_m(t)\,(w_m(t)-1) + \rho_{n,m}(t)\,\frac{\sigma_n(t)}{\sigma_m(t)}w_n(t)\,w_m(t)\right) + \mathcal{O}\left(\gamma^2(t)\right).\end{aligned}$$

Hence, for any real constants (A, B) we have

$$\frac{\sigma_n(t)}{\bar{\sigma}_n(t)}\frac{\sigma_m(t)}{\bar{\sigma}_m(t)}\left(\gamma_n(t)\,A + \gamma_m(t)\,B\right) = \gamma_n(t)\,A + \gamma_m(t)\,B + \mathcal{O}\left(\gamma^2(t)\right).$$

Proof. The second result is a direct consequence of the first, which follows from a standard Taylor expansion in γ. ☐

We now apply Eq. (70) to the cross term between P_n and P_m. This gives us

$$\begin{aligned}\rho_{n,m}(t)\sigma_n(t)\,\sigma_m(t) \to& \rho_{n,m}(t)\,\sigma_n(t)\,\sigma_m(t)\\ &+ \rho_{n,m}(t)\,\sigma_n(t)\,\sigma_m(t)\left(\gamma_n(t)\,w_n(t)\,(w_n(t)-1) + \gamma_m(t)\,w_m(t)\,(w_m(t)-1)\right)\\ &+ \left(\gamma_n(t)\,\sigma_m^2(t) + \gamma_m(t)\,\sigma_n^2(t)\right)w_n(t)\,w_m(t) + \mathcal{O}\left(\gamma^2(t)\right).\end{aligned}$$

Since the goal of this Appendix is to derive an expression for the new market correlation, denoted $\bar{\rho}_{n,m}$, we need to evaluate

$$\begin{aligned}\bar{\rho}_{n,m}(t) =& \rho_{n,m}(t)\,\frac{\sigma_n(t)\,\sigma_m(t)}{\bar{\sigma}_n(t)\,\bar{\sigma}_m(t)}\left(1 + \gamma_n(t)\,w_n(t)\,(w_n(t)-1) + \gamma_m(t)\,w_m(t)\,(w_m(t)-1)\right)\\ &+ \frac{\sigma_n(t)\,\sigma_m(t)}{\bar{\sigma}_n(t)\,\bar{\sigma}_m(t)}\left(\gamma_n(t)\,\frac{\sigma_m(t)}{\sigma_n(t)} + \gamma_m(t)\,\frac{\sigma_n(t)}{\sigma_m(t)}\right)w_n(t)\,w_m(t).\end{aligned}$$

A straightforward application of Lemma A.1 gives the final result.

$$\bar{\rho}_{n,m}(t) = \rho_{n,m}(t) + \left(1 - \rho_{n,m}^2(t)\right)\left(\gamma_n(t)\,\frac{\sigma_m(t)}{\sigma_n(t)} + \gamma_m(t)\,\frac{\sigma_n(t)}{\sigma_m(t)}\right)w_n(t)\,w_m(t) + \mathcal{O}\left(\gamma^2(t)\right).$$

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Kelly trading and marketing equilibrium

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We show that the Kelly framework is the natural multi-period extension of the one-period meanvariance model of Markowitz. Any allocation on the instantaneous Kelly efficient frontier can be reached by trading in the bank account and a particular mutual fund consisting of risky assets only. However, different to the mean-variance model there is an upper bound on the instantaneous Kelly efficient frontier. Any allocation surpassing this bound, called the optimal Kelly strategy, is known to be inefficient. We use the optimal Kelly strategy to deduce an expression for the instantaneous covariance matrix that results in market equilibrium. We show that this equilibrium is stable in the sense that any optimal Kelly trader, and most Kelly traders, will force the market covariance matrix towards equilibrium. We further investigate the role of a central bank setting the short-term interest rate. We show that an equilibrium rate exists but we argue that there is no market mechanism that will force the interest rate to this level. This motivates the short-term interest rate to be actively managed by the central bank.

KEYWORDS: Portfolio theory, Kelly criterion, Market equilibrium, Central bank rate

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