RESEARCH ARTICLE



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Kelly trading and option pricing

Hans-Peter Bermin^{1,2}

 ¹Knut Wicksell Centre for Financial Studies, Lund University, Lund, Sweden
 ²Hilbert Capital, Valletta, Malta

Correspondence

Hans-Peter Bermin, Knut Wicksell Centre for Financial Studies, Lund University, Box 7080, SE-220 07 Lund, Sweden. Email: hans-peter.bermin@nek.lu.se

Magnus Holm²

Abstract

In this paper we show that a Kelly trader is indifferent to trade a derivative if and only if the no-arbitrage price is uniquely given by the minimal martingale measure price, thus providing a natural selection mechanism for option pricing in incomplete markets. We also show that the unique Kelly indifference price results in market equilibrium in the sense that no Kelly trader can improve the magnitude of his instantaneous Sharpe ratio, by trading the derivative, given the actions of the other market participants.

KEYWORDS

Hansen-Jagannathan bound, incomplete markets, Kelly indifference price, minimal martingale measure, option pricing

JEL CLASSIFICATION G11, G12, G13

1 | INTRODUCTION

In this paper we apply the optimal growth theory of Kelly (1956) and Latané (1959) to option pricing in incomplete markets. When the market is the incomplete absence of arbitrage is not enough to determine a unique price for a derivative. Instead, the absence of arbitrage produces a range of possible prices that are all consistent with no-arbitrage. Which price to pick is often a matter of taste though some candidates have proven more popular than others. In this paper we give special attention to the arbitrage-free price obtained by using the so-called minimal martingale measure of Föllmer and Schweizer (1991). In essence, the minimal martingale measure approach says that only the risk inherent in the primary assets has a nonzero market price. Any other risk component has a market price of zero, which in particular applies to derivatives not spanned by the assets at hand.

The range of arbitrage-free prices is in general considered far too wide to be of practical relevance. This led Cochrane and Saá Requejo (2000) to define the concept of no-good-deal prices. Similar to no-arbitrage pricing the no-good-deal pricing generates a range of prices all consistent with no good-deals. The price intervals are indexed with a real positive parameter and form an increasing sequence within the no-arbitrage price range. In the limit as the index approaches zero we are left with a unique price given by the minimal martingale measure price, while in the limit as the index approaches infinity the no-good-deal price range essentially coincides with the no-arbitrage price range. The mathematical properties of the associated optimization problem leading to the upper and lower no-good-deal bounds have formally been characterized in Björk and Slinko (2006). In short one can view the no-good-deal bounds as an arbitrage-free price with an additional constraint on the instantaneous Sharpe ratio of the derivative. The constraint is, however, enforced via the market price of risk process using the bounds derived in Hansen and Jagannathan (1991). By

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applying a reasonable bound for the instantaneous Sharpe ratio of the derivative the no-good-deal bound tightens to the extent that it becomes practical relevant, see Cochrane and Saá Requejo (2000) for further details.

What makes the optimal growth theory ideal to use in this setting is that these trading strategies have a maximal instantaneous Sharpe ratio. Hence, they always attain the Hansen–Jagannathan bounds. This allows us to quantify the additional gain a trader can make by adding a derivative to his existing portfolio. We call such trading strategies Kelly strategies using the terminology in Bermin and Holm (2019), while other sources, like, MacLean et al. (1992) and Davis and Lleo (2013), essentially denote them fractional Kelly strategies. The contribution of this paper is to show that for each no-good-deal price range, identified by the fixed index parameter, a Kelly investor is indifferent to trade the derivative if and only if the price coincides with the minimal martingale measure price. In this situation, the Kelly trader cannot improve his instantaneous Sharpe ratio by adding the derivative to the existing portfolio. Hence, our approach is similar in spirit to the utility indifference pricing of Davis (1997), although no specific utility specification of the investors is required. We further show that the only price within the no-arbitrage price range that is consistent with a Kelly trader being indifferent to trade the derivative is the minimal martingale measure price. To strengthen our result we also show that this price corresponds to the market being in a trading equilibrium. Hence, similar to the original paper of Black and Scholes (1973) we use equilibrium argument to derive a unique price for the derivative. In this regard, we extend their approach to an incomplete market setting.

2 | MODELING THE MARKET

We consider a capital market consisting of a bank account *B* and a number of assets $P = (P_1, ..., P_N)'$. An asset related to a dividend-paying stock is seen as a fund with the dividends re-invested. All assets are assumed to be adapted stochastic processes living on a filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$, where $\mathbb{F} = \{\mathcal{F}(t) : t \ge 0\}$ is a right-continuous increasing family of σ -algebras such that $\mathcal{F}(0)$ contains all the \mathbb{P} -null sets of \mathcal{F} . As usual we think of the filtration \mathbb{F} as the carrier of information. We assume that all stochasticity is generated by an *M*-dimensional standard Brownian motion *W*, where $M \ge N$, and we identify the filtration \mathbb{F} with the \mathbb{P} -augmentation of the natural filtration of *W*. Furthermore, we write

$$W(t) = \begin{pmatrix} W^1(t) \\ W^2(t) \end{pmatrix},$$

where $W^1 = (W_1, ..., W_N)'$ contains the first *N* components of *W*, while $W^2 = (W_{N+1}, ..., W_M)'$ contains the remaining ones. The reason for this decomposition is that we want to separate the Brownian components driving the stochasticity of the asset prices from those needed to make the model parameters measurable.

We assume that the bank account is locally risk-free with

$$d\log B(t) = r(t)dt,$$

and thus fully determined by an \mathbb{F} -adapted interest rate process r. Regarding the risky assets we set

$$d\log P_n(t) = \mu_n(t)dt + \Sigma'_n(t)dW^1(t), \quad 1 \le n \le N,$$

where the logarithmic drift $\mu = (\mu_1, ..., \mu_N)'$ and the volatility matrix $\Sigma = (\Sigma_1, ..., \Sigma_N)'$ are arbitrary \mathbb{F} -adapted processes. We note that each volatility vector process Σ_n , $1 \le n \le N$, takes values in \mathbb{R}^N and throughout this paper we assume that Σ is a.s. invertible. We also let $\sigma = (||\Sigma_1||, ..., ||\Sigma_N||)'$ be the vector of real-valued asset volatilities and write $\sigma_{\text{diag}} = \text{diag}(\sigma)$ for the associated diagonal matrix. The instantaneous asset–asset covariance matrix *V* can then be expressed in terms of the corresponding instantaneous correlation matrix ρ according to

$$V(t) = \Sigma(t)\Sigma'(t) = \sigma_{\text{diag}}(t)\rho(t)\sigma_{\text{diag}}(t).$$
⁽¹⁾

We further introduce the instantaneous Sharpe ratios $s = (s_1, ..., s_N)'$, as defined in Nielsen and Vassalou (2004), according to

$$s_n(t) = \frac{1}{2}\sigma_n(t) + \frac{\mu_n(t) - r(t)}{\sigma_n(t)}, \quad 1 \le n \le N.$$
 (2)

To better understand the separation of the Brownian motion W into the components W^1 and W^2 we introduce a number of nontradable indices $I = (I_1, ..., I_{M-N})'$ according to

$$dI(t) = \gamma(t)dt + \Gamma(t)dW^{1}(t) + \tilde{\Gamma}(t)dW^{2}(t),$$

where the model parameters of the indices are taken to be arbitrary \mathbb{F} -adapted processes. More specifically, we set $\gamma = (\gamma_1, ..., \gamma_{M-N})'$, $\Gamma = (\Gamma_1, ..., \Gamma_{M-N})'$, and $\tilde{\Gamma} = (\tilde{\Gamma}_1, ..., \tilde{\Gamma}_{M-N})'$, such that the components of Γ take values in \mathbb{R}^N while the components of $\tilde{\Gamma}$ take values in \mathbb{R}^{M-N} . Throughout this paper we allow the indices to represent, say, the interest rate *r* or the drift of a particular asset μ_n or the volatility σ_n , according to our choice. As will be shown later the purpose of introducing the indices is to be able to work within an extended Markovian state space.

In addition to the tradable assets (B, P) and the nontradable indices *I* we introduce one derivative asset whose price process is denoted by Π . This financial contract comes with an expiry date, *T*, at which time the contract pays the amount

$$\Pi(T) = \Phi(P(T)) \quad \text{a.s.} \tag{3}$$

for some deterministic function $\Phi : \mathbb{R}^N_+ \to \mathbb{R}_+$. To be precise a derivative of this form is said to be European and in this paper we solely focus on these types. We do stress, however, that more complex derivatives can often be treated in much a similar way. We now assume that the price of the derivative evolves according to

$$d\log \Pi(t) = \mu_{\pi}(t)dt + \sum_{\pi}'(t)dW^{1}(t) + \widetilde{\Sigma}_{\pi}'(t)dW^{2}(t), \quad 0 \le t \le T$$
(4)

for some \mathbb{F} -adapted processes ($\mu_{\pi}, \Sigma_{\pi}, \tilde{\Sigma}_{\pi}$). Similar to Equation (2) we also define the instantaneous Sharpe ratio of the derivative

$$s_{\pi}(t) = \frac{1}{2}\sigma_{\pi}(t) + \frac{\mu_{\pi}(t) - r(t)}{\sigma_{\pi}(t)}, \quad \sigma_{\pi}^{2}(t) = \|\Sigma_{\pi}(t)\|^{2} + \|\tilde{\Sigma}_{\pi}(t)\|^{2}.$$

Finally, we let $\rho_{\pi} = (\rho_{\pi,1}, ..., \rho_{\pi,N})'$ denote the instantaneous correlation between the derivative and each of the risky assets according to

$$\rho_{\pi,n}(t) = \frac{\Sigma'_{\pi}(t)\Sigma_n(t)}{\sigma_{\pi}(t)\sigma_n(t)}, \quad 1 \le n \le N.$$
(5)

With the notation for the tradable assets and the nontradable indices put in place we now turn our attention to trading. An investor can invest in the tradable assets and for the sake of simplicity we assume that there are no transaction fees, that short-selling is allowed, that trading takes place continuously in time, and that the investor's trading activity does not impact the asset prices. We define a trading strategy as an \mathbb{F} -predictable process $q = (q_0, q_1, ..., q_{N+1})'$ representing the number of shares held in each asset. However, in many situations it is often more convenient to define the trading strategy as being proportional to the portfolio wealth *X*. For this reason we introduce $w = (w_1, ..., w_N)'$ and w_{N+1} according to

$$w_n(t)X(t) = q_n(t)P_n(t), \quad w_{N+1}(t)X(t) = q_{N+1}(t)\Pi(t).$$

To analyze the performance of the portfolio process we impose the restriction that, when rebalancing the portfolio, money can neither be injected nor withdrawn. Such trading strategies are said to be self-financing and imply that the portfolio process evolves according to

$$\frac{dX(t)}{X(t)} = w_0(t)\frac{dB(t)}{B(t)} + \sum_{n=1}^N w_n(t)\frac{dP_n(t)}{P_n(t)} + w_{N+1}(t)\frac{d\Pi(t)}{\Pi(t)},$$

where we have set $w_0 = 1 - \sum_{n=1}^{N+1} w_n$. By using the quadratic covariation process, we now define the volatility process according to

$$\sigma_X^2(t) = \frac{1}{X^2(t)} \frac{d}{dt} [X, X](t) = \frac{d}{dt} [\log X, \log X](t).$$

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The evolution of the logarithmic portfolio process further takes the explicit form

$$d\log X(t) = \mu_X(t)dt + (w'(t)\Sigma(t) + w_{N+1}(t)\sum_{\pi}'(t))dW^1(t) + w_{N+1}(t)\widetilde{\Sigma}_{\pi}'(t)dW^2(t),$$

where

$$\mu_X(t) = r(t) + w'(t)\sigma_{\text{diag}}(t)s(t) + w_{N+1}(t)\sigma_{\pi}(t)s_{\pi}(t) - \frac{1}{2}\sigma_X^2(t),$$

$$\sigma_X^2(t) = w'(t)V(t)w(t) + w_{N+1}^2(t)\sigma_{\pi}^2(t) + 2w_{N+1}(t)\sigma_{\pi}(t)w'(t)\sigma_{\text{diag}}(t)\rho_{\pi}(t)$$

such that the corresponding instantaneous Sharpe ratio equals

$$s_X(t) = \frac{1}{2}\sigma_X(t) + \frac{\mu_X(t) - r(t)}{\sigma_X(t)}.$$
(6)

Of course, in order for the instantaneous Sharpe ratio of the portfolio to be well defined we must impose integrability conditions on the model parameters and on the trading strategy. We also stress that an investor can only engage in trading as long as his wealth is positive. This motivates the definition below.

Definition 2.1. Given a trading horizon *T* and initial capital $X(0) \ge 0$. A self-financing trading strategy is said to be admissible if $X(t) \ge 0$, $0 \le t \le T$, a.s. and

$$\mathbb{P}\left(\int_0^T \left(|\mu_X(t) - r(t)| + \sigma_X^2(t)\right) dt < \infty\right) = 1$$

However, this is not enough to ensure the instantaneous Sharpe ratio of the portfolio to be well defined. If, say, we can find a trading strategy such that the volatility $\sigma_X = 0$ a.s., over some time interval, the instantaneous Sharpe ratio might still explode. For this reason we must impose the additional requirement of no-arbitrage in the capital market and in Section 3 we briefly recap the main elements.

3 | NO-ARBITRAGE PRICING

The absence of arbitrage is a very simple and intuitive concept yet its uses in continuous time finance are at times rather technical. The purpose of this section is to summarize the established theory while keeping the technicalities at a minimum. To do so we have chosen to work with admissible trading strategies that ensure the portfolio process to be positive. This implies that bankruptcy is an absorbing state, see Remark 3.3.4 in Karatzas and Shreve (1999), which makes a lot of sense from an economic point of view. However, much of the no-arbitrage results can be applied to a weaker notion of admissible trading strategies. For instance, it is often sufficient to assume that the portfolio process is a.s. bounded from below by some real constant. The problem with such extensions though lies in the practical justifications.

An arbitrage opportunity is a trading strategy for which invested money cannot be lost but profits can be made. Typically such strategies evolve shortening one asset and buying another. Another way to express an arbitrage opportunity is to relate it to the locally risk-free bank account and this is the path we choose.

Definition 3.1. Given a trading horizon *T* and initial capital $X(0) \ge 0$. An admissible trading strategy is said to be an *arbitrage opportunity* if

$$\mathbb{P}\left(\frac{X(T)}{X(0)} \ge \frac{B(T)}{B(0)}\right) = 1, \quad \mathbb{P}\left(\frac{X(T)}{X(0)} > \frac{B(T)}{B(0)}\right) > 0.$$

A capital market where no-arbitrage opportunities exist is said to be arbitrage-free.

To state what it takes to rule out arbitrage opportunities and prepare the ground for future applications it is beneficial to introduce a number of additional concepts.

Definition 3.2. We say that \mathbb{R}^{M} -valued \mathbb{F} -adapted process θ belongs to the space \mathcal{I}_{T} if

$$\mathbb{P}\left(\int_0^T \|\theta(t)\|^2 dt < \infty\right) = 1.$$

For any $\mathbb{R}^{K \times M}$ -valued \mathbb{F} -adapted process A, where K is an arbitrary integer, we further define the subsets

$$\mathcal{K}_T(A) = \{ \theta \in \mathcal{I}_T : \theta(t) \in \text{kernel}A(t), \forall t \in [0, T] \text{ a.s. } \}, \\ \mathcal{K}_T^{\perp}(A) = \{ \theta \in \mathcal{I}_T : \theta(t) \in \text{range}A'(t), \forall t \in [0, T] \text{ a.s. } \}.$$

The next result shows that the subsets $\mathcal{K}_T(A)$ and $\mathcal{K}_T^{\perp}(A)$ are orthogonal and that the decomposition can be computed given the pseudo-inverse A^+ of A.

Lemma 3.3. Given an $\mathbb{R}^{N \times M}$ -valued \mathbb{F} -adapted process A. Every process $\theta \in \mathcal{I}_T$ admits a unique orthogonal decomposition

$$\theta(t) = \theta_{\parallel}(t) + \theta_{\perp}(t), \quad \theta_{\parallel} \in \mathcal{K}_T(A), \quad \theta_{\perp} \in \mathcal{K}_T^{\perp}(A).$$

In terms of the pseudo-inverse A^+ of A the decomposition takes the form

$$\theta_{\parallel}(t) = (I - A^{+}(t)A(t))\theta(t), \quad \theta_{\perp}(t) = A^{+}(t)A(t)\theta(t)$$

Proof. We first note that the decomposition is valid since $\theta_{\parallel} + \theta_{\perp} = \theta$. Next, from the definition of a pseudoinverse we know that $AA^+A = A$ and $(A^+A)' = A^+A$. This shows that θ_{\parallel} is in the kernel of A and that

$$\theta'_{\perp}(t)\theta_{\parallel}(t) = \theta'(t)A^{+}(t)A(t)(I - A^{+}(t)A(t))\theta(t) = 0.$$

Hence, the decomposition is orthogonal. Finally, since the pseudo-inverse further satisfies $A^+ = A'(AA')^+$ it follows that θ_{\perp} is in the range of A'. For additional details on pseudo-inverses, see Ben-Israel and Greville (2003).

We now apply these results to the capital market (B, P, Π) consisting of the bank account, the risky assets and the derivative. The results below are taken from Karatzas and Shreve (1999) where additional information is provided.

Proposition 3.4. In an arbitrage-free capital market there exists a process $\theta \in I_T$, known as the market price of risk, such that

$$\begin{pmatrix} \sigma_{\text{diag}}(t)s(t) \\ \sigma_{\pi}(t)s_{\pi}(t) \end{pmatrix} = \begin{pmatrix} \Sigma(t) & \mathbf{0}_{N \times M - N} \\ \Sigma'_{\pi}(t) & \widetilde{\Sigma}'_{\pi}(t) \end{pmatrix} \theta(t).$$

We let Θ_T denote the collection of such processes $\theta \in \mathcal{I}_T$.

Proof. Let us introduce the notation

$$y(t) = \begin{pmatrix} \sigma_{\text{diag}}(t)s(t) \\ \sigma_{\pi}(t)s_{\pi}(t) \end{pmatrix}, \quad A(t) = \begin{pmatrix} \Sigma(t) & \mathbf{0}_{N \times M - N} \\ \Sigma'_{\pi}(t) & \widetilde{\Sigma}'_{\pi}(t) \end{pmatrix},$$

such that we can write the equation to be proved as $y = A\theta$. We want to show that the absence of arbitrage opportunities implies that y is in the range of A, that is, $y \in \mathcal{K}_T^{\perp}(A')$, but this is equivalent to show that there exist arbitrage opportunities if $y \in \mathcal{K}_T(A')$. Now consider the self-financing trading strategy defined by

$$\hat{w}(t) = \begin{pmatrix} w(t) \\ w_{N+1}(t) \end{pmatrix} = \frac{\epsilon}{\|y(t)\|^2} y(t), \quad \epsilon > 0.$$

Hence, for $y \in \mathcal{K}_T(A')$ it follows that $\hat{w} \in \mathcal{K}_T(A')$, which further implies that the portfolio process satisfies

$$d\log X(t) = (r(t) + \epsilon)dt \Leftrightarrow \frac{X(T)}{X(0)} = \frac{B(T)}{B(0)}e^{\epsilon T}.$$

This shows that \hat{w} is an arbitrage opportunity, according to Definition 3.1, which concludes the proof.

While the instantaneous Sharpe ratio processes are uniquely defined for each security the market price of risk process is in general not. Below, we illustrate this point using the decomposition $\theta^1 = (\theta_1, ..., \theta_N)'$ and $\theta^2 = (\theta_{N+1}, ..., \theta_M)'$.

Theorem 3.5. Let the capital market be arbitrage-free. Then any market price of risk process

$$\theta(t) = \begin{pmatrix} \theta^1(t) \\ \theta^2(t) \end{pmatrix} \in \Theta_T$$

satisfies

$$\theta^{1}(t) = \Sigma^{-1}(t)\sigma_{\text{diag}}(t)s(t), \quad \theta^{2}(t) = \theta^{2}_{\perp}(t) + \theta^{2}_{\parallel}(t)$$

where θ_{\perp}^2 and θ_{\parallel}^2 are orthogonal and equal to

$$\begin{aligned} \theta_{\perp}^{2}(t) &= \|\widetilde{\Sigma}_{\pi}(t)\|^{-2}\widetilde{\Sigma}_{\pi}(t) \Big(\sigma_{\pi}(t)s_{\pi}(t) - \Sigma_{\pi}'(t)\theta^{1}(t)\Big), \\ \theta_{\parallel}^{2}(t) &= \Big(I - \|\widetilde{\Sigma}_{\pi}(t)\|^{-2}\widetilde{\Sigma}_{\pi}(t)\widetilde{\Sigma}_{\pi}'(t)\Big)\phi(t), \end{aligned}$$

for some arbitrary \mathbb{F} -adapted process ϕ taking values in \mathbb{R}^{M-N} .

Proof. The solution for θ^1 follows directly from Proposition 3.4. With the aid of Lemma 3.3 we further introduce the decomposition $\theta^2 = \theta_{\perp}^2 + \theta_{\parallel}^2$, such that $\theta_{\perp}^2 \in \text{range}\tilde{\Sigma}_{\pi}$ and $\theta_{\parallel}^2 \in \text{kernel}\tilde{\Sigma}'_{\pi}$. Hence, θ_{\perp}^2 and θ_{\parallel}^2 are orthogonal. Straightforward calculations now verify that

$$\sigma_{\pi}(t)s_{\pi}(t) = \Sigma_{\pi}'(t)\theta^{1}(t) + \widetilde{\Sigma}_{\pi}'(t)\theta^{2}(t),$$

which completes the proof.

The sole existence of a market price of risk process $\theta \in \Theta_T$ is enough to rule out a multitude of arbitrage opportunities. Unfortunately, though, it does not rule out all possible arbitrage opportunities. The reason is that a market price of risk process $\theta \in \Theta_T$ can, in a mathematical sense, be too wild, leading to what can be considered as pathological results. For this reason we introduce the additional notation.

Definition 3.6. For $\theta \in \Theta_T$, we introduce the local martingale

$$Z(t) = \exp\left(-\frac{1}{2}\int_0^t \|\theta(s)\|^2 \, ds - \int_0^t \theta'(s) \, dW(s)\right), \quad 0 \le t \le T.$$

We further let $\Theta_T^{\mathcal{M}} = \{\theta \in \Theta_T : \mathbb{E}[Z(T)] = 1\}$ denote the subclass of processes for which Z is a martingale.

The process *Z*/*B* is often referred to as the stochastic discount factor and consequently $\|\theta\|$ represents the associated volatility. The importance of the subspace $\Theta_T^{\mathcal{M}}$ was first highlighted in Harrison and Pliska (1981) as explained below.

Proposition 3.7. There exist no-arbitrage opportunities if the market price of risk process $\theta \in \Theta_T^M$.

Proof. While credit is due to Harrison and Pliska (1981) we also refer to Karatzas and Shreve (1999), Theorem 1.4.2, for a comprehensive proof.

In order for θ to belong to the subspace $\Theta_T^{\mathcal{M}}$ the norm $\|\theta\|^2 = \|\theta^1\|^2 + \|\theta^2\|^2$ must satisfy additional integrability conditions. Leaving aside the technicalities we notice, using Theorem 3.5, that the norm of θ^2 satisfies

$\|\theta^{2}(t)\|^{2} = \|\theta^{2}_{\perp}(t)\|^{2} + \|\theta^{2}_{\parallel}(t)\|^{2} \ge \|\theta^{2}_{\perp}(t)\|^{2}.$

Hence, the norm $\|\theta\|$ is minimal when the arbitrary process ϕ equals the zero-vector, in which case we introduce an upper bound on the instantaneous Sharpe ratio of the derivative according to

$$\hat{c} = \sup \left\{ c \ge 0 : \theta \in \Theta_T^{\mathcal{M}}, \, \theta^1 = \Sigma^{-1} \sigma_{\text{diag}} s, \, \theta^2 = c \| \widetilde{\Sigma}_{\pi} \|^{-1} \widetilde{\Sigma}_{\pi} \right\}.$$

$$\tag{7}$$

From here onward we assume that the market price of risk process $\theta \in \Theta_T^{\mathcal{M}}$, thereby ensuring the capital market to be arbitrage-free. For each choice of θ we then define the equivalent probability measure $\mathbb{P}_{\theta}(A) = \mathbb{E}[Z_{\theta}(T)\mathbf{1}\{A\}], A \in \mathcal{F}(T)$, such that

$$V_{\theta}(t) = W(t) + \int_0^t \theta(s) \, ds, \quad 0 \le t \le T,$$

is a Pe-Brownian motion according to the Girsanov theorem. A direct application of Itô's lemma now yields

$$d\frac{\Pi(t)}{B(t)} = \frac{\Pi(t)}{B(t)} \Big(\Sigma'_{\pi}(t) dV^{1}_{\theta}(t) + \widetilde{\Sigma}'_{\pi}(t) dV^{2}_{\theta}(t) \Big),$$

where V_{θ}^1 and V_{θ}^2 are defined analogously to W^1 and W^2 . Hence, with respect to the \mathbb{P}_{θ} -measure, the process Π/B is a nonnegative local martingale and thus a supermartingale. If we further assume that the derivative payoff, as expressed in Equation (3), is sufficiently integrable for Π/B to be a martingale and that the additional indices are chosen in such a way as to generate a Markovian system, we may define the arbitrage-free price as below.

Definition 3.8. The no-arbitrage price of the derivative is defined by

$$\frac{\Pi(t)}{B(t)} = \mathbb{E}_{\theta} \left[\frac{\Phi(P(T))}{B(T)} \middle| P(t), I(t) \right], \quad \theta \in \Theta_T^{\mathcal{M}},$$

where the expectation is to be computed subject to the dynamics

$$d\log P_n(t) = \left(r(t) - \frac{1}{2}\sigma_n^2(t)\right)dt + \Sigma'_n(t)dV_\theta^1(t), \quad 1 \le n \le N,$$

$$dI(t) = (\gamma(t) - \Gamma(t)\theta^1(t) - \widetilde{\Gamma}(t)\theta^2(t))dt + \Gamma(t)dV_\theta^1(t) + \widetilde{\Gamma}(t)dV_\theta^2(t)$$

By varying θ , or more precisely the free parameter θ^2 , we see that there exists an interval $[\min_{\theta^2}\Pi, \max_{\theta^2}\Pi]$ of prices that are all consistent with no-arbitrage. We further observe that only if $\|\tilde{\Gamma}_i\| = 0$, for all *i*, can we categorically claim that the arbitrage-free price of the derivative is unique. The reason is simply that, in this case, the terminal payoff can be perfectly replicated by trading in the primary assets as originally proved in Merton (1973b). In the general case, referred to as an incomplete market, a unique price can sometimes be derived given assumptions on the specific structure of the component θ^2 . However, such assumptions generally lead to the price of the derivative approaching the upper no-arbitrage pricing bound, see Karatzas et al. (1991) for details. It is fair to say that the no-arbitrage pricing interval is too wide to be of any practical relevance and that derivative prices observed in the market are typically not in line with the upper no-arbitrage pricing bound. To restrict the noarbitrage pricing interval, in search for market-aliened derivative prices, we may introduce a bound on the instantaneous Sharpe ratio of the derivative. This gives rise to a pricing methodology known as no-good-deal pricing, which brings us to Section 4.

4 | NO-GOOD-DEAL PRICING

The concept of no-good-deal pricing was first introduced in Cochrane and Saá Requejo (2000). The arguments are based on the historical observation that Sharpe ratios, for most assets, typically fluctuates around the value of 0.5. Some assets have a higher ratio, some have a lower, so a fund manager combining assets cannot expect to generate a portfolio with an arbitrarily high Sharpe ratio. In fact a portfolio having a Sharpe ratio above one is considered very good, while a

Sharpe ratio above two is considered outstanding. To put these numbers in perspective we recall a result in Bermin and Holm (2019) showing that the instantaneous Sharpe ratio of any portfolio must be finite if the capital market is arbitrage-free. Hence, there is a huge difference, in numerical value of the Sharpe ratio, between a good deal and an arbitrage opportunity. The no-good-deal pricing can be thought of as imposing a bound on the instantaneous Sharpe ratio of the derivative. However, for technical reasons it turns out to be much easier to impose a bound directly on the market price of risk component θ^2 . The connection between the market price of risk process (or equally the volatility of the stochastic discount factor) and the instantaneous Sharpe ratio is known as the Hansen–Jagannathan bound and was first derived in Hansen and Jagannathan (1991). In our setting their result takes the form:

Proposition 4.1. A portfolio that can only trade in the primary assets has an instantaneous Sharpe ratio that satisfies

$$s_X^2(t) \le s'(t)\rho^{-1}(t)s(t),$$

while a portfolio that can trade in the primary assets and the derivative satisfies

$$s_X^2(t) \le s'(t)\rho^{-1}(t)s(t) + \|\tilde{\Sigma}_{\pi}(t)\|^{-2} \Big(\sigma_{\pi}(t)s_{\pi}(t) - \Sigma'_{\pi}(t)\theta^1(t)\Big)^2.$$

Proof. Suppose first that the derivative is excluded from the portfolio, that is, $w_{N+1} = 0$. By the use of Proposition 3.4 the instantaneous Sharpe ratio of the portfolio, Equation (6), then equals

$$s_X(t) = \frac{w'(t)\sigma_{\text{diag}}(t)s(t)}{\sigma_X(t)} = \frac{w'(t)\Sigma(t)\theta^1(t)}{\sigma_X(t)}.$$

Hence, by applying the Cauchy-Schwartz inequality we obtain

$$s_X^2(t) \le \frac{\|\Sigma'(t)w(t)\|^2 \|\theta^1(t)\|^2}{\sigma_X^2(t)} = \|\theta^1(t)\|^2.$$

Next, let us consider the case when the derivative is included. Again, using Proposition 3.4, we express the instantaneous Sharpe ratio according to

$$s_X(t) = \frac{w'(t)\sigma_{\text{diag}}(t)s(t) + w_{N+1}(t)\sigma_{\pi}(t)s_{\pi}(t)}{\sigma_X(t)}$$
$$= \frac{1}{\sigma_X(t)} \Big(w'(t)\Sigma(t) + w_{N+1}(t)\Sigma'_{\pi}(t)w_{N+1}(t)\widetilde{\Sigma}'_{\pi}(t) \Big) \Big(\begin{pmatrix} \theta^1(t) \\ \theta^2(t) \end{pmatrix} \Big).$$

A second application of Cauchy-Schwartz inequality now yields

$$s_X^2(t) \le \|\theta(t)\|^2 = \|\theta^1(t)\|^2 + \|\theta^2(t)\|^2.$$

We finally apply Theorem 3.5 to obtain $s_X^2 \le \|\theta^1\|^2 + \|\theta_{\perp}^2\|^2 + \|\theta_{\parallel}^2\|^2$ and compute

$$\begin{split} \|\theta^{1}(t)\|^{2} &= s'(t)\rho^{-1}(t)s(t), \\ \|\theta^{2}_{\perp}(t)\|^{2} &= \left\|\widetilde{\Sigma}_{\pi}(t)\right\|^{-2} \left(\sigma_{\pi}(t)s_{\pi}(t) - \Sigma'_{\pi}(t)\theta^{1}(t)\right)^{2}, \\ \|\theta^{2}_{\parallel}(t)\|^{2} &= \phi'(t) \left(I - \|\widetilde{\Sigma}_{\pi}(t)\|^{-2}\widetilde{\Sigma}_{\pi}(t)\widetilde{\Sigma}'_{\pi}(t)\right) \phi(t). \end{split}$$

Since the Hansen–Jagannathan bound must hold for any vector process ϕ it particularly holds for the zero-vector.

Having derived bounds for the instantaneous Sharpe ratio of an arbitrary trading strategy we now focus on the bounds for the derivative.

Corollary 4.2. The instantaneous Sharpe ratio of the derivative is bounded from above and satisfies

$$s_{\pi}^{2}(t) \leq s'(t)\rho^{-1}(t)s(t) + \|\widetilde{\Sigma}_{\pi}(t)\|^{-2} \Big(\sigma_{\pi}(t)s_{\pi}(t) - \Sigma'_{\pi}(t)\theta^{1}(t)\Big)^{2}.$$

Proof. This is a special case of Proposition 4.1, with w equal to the zero-vector and $w_{N+1} = 1$.

One notices that the upper bound of the instantaneous squared Sharpe ratio of the derivative is related to the norm of the market price of risk component θ_{\perp}^2 . However, since this term depends on the Sharpe ratio s_{π} we are faced with an inequality to solve. An alternative approach is the one taken in Cochrane and Saá Requejo (2000) when defining the no-good-deal price.

Definition 4.3. The no-good-deal price of the derivative is defined by

$$\frac{\Pi_{c}(t)}{B(t)} = \mathop{\mathbb{E}}_{\|\theta^{2}\| \le c} \left[\frac{\Phi(P(T))}{B(T)} \middle| P(t), I(t) \right], \quad \theta \in \Theta_{T}^{\mathcal{M}}, \quad c \in \mathbb{R}_{+},$$

where the expectations are to be computed subject to the dynamics

$$d \log P_n(t) = \left(r(t) - \frac{1}{2}\sigma_n^2(t)\right)dt + \Sigma'_n(t)dV_\theta^1(t), \quad 1 \le n \le N,$$

$$dI(t) = \left(\gamma(t) - \Gamma(t)\theta^1(t) - \widetilde{\Gamma}(t)\theta^2(t)\right)dt + \Gamma(t)dV_\theta^1(t) + \widetilde{\Gamma}(t)dV_\theta^2(t).$$

From the Definition above we see that the no-good-deal price interval is increasing with respect to the parameter c as

$$[\min_{\theta^2} \Pi_{c_1}(t), \max_{\theta^2} \Pi_{c_1}(t)] \subseteq [\min_{\theta^2} \Pi_{c_2}(t), \max_{\theta^2} \Pi_{c_2}(t)], \quad 0 \le c_1 \le c_2.$$

Moreover, the upper bound for the instantaneous squared Sharpe ratio of the derivative takes the simple form

$$s_{\pi}^{2}(t) \leq \|\theta^{1}(t)\|^{2} + \|\theta^{2}(t)\|^{2} \leq s'(t)\rho^{-1}(t)s(t) + c^{2}.$$

However, although the definition of the no-good-deal price looks fairly innocent it is far so. The reason is, as mentioned earlier, that θ^2 generally contains information about the derivative price process via s_{π} . Nevertheless, as shown in Cochrane and Saá Requejo (2000) and in particular Björk and Slinko (2006), this definition leads to a nonstandard Hamilton–Jacobi–Bellman equation which allows us to, at least, compute the upper and lower no-good-deal bounds.

Proposition 4.4. The bounds of the no-good-deal price Π_c , $c \ge 0$, correspond to

$$\theta^2(t) = \pm c \|\tilde{\Sigma}_{\pi}(t)\|^{-1} \tilde{\Sigma}_{\pi}(t).$$

Proof. The proof follows from Björk and Slinko (2006) and is further explained in the appendix.

There are two important conclusions to be drawn from this result. First, we see from Equation (7) that the noarbitrage bounds can be obtained from the no-good-deal bounds when $c = \hat{c}$ and second, we see that there is still an intricate recursive relationship between Π_c and θ^2 , through the volatility component $\tilde{\Sigma}_{\pi}$. Leaving aside the details about the explicit calculations we observe that only when c = 0, in which case θ^2 equals the zero-vector, can we compute the no-good-deal price in a straightforward way. In this special case the no-good-deal price is unique and corresponds to, what we call, the minimal martingale measure price. The minimal martingale measure was originally introduced in Föllmer and Schweizer (1991) in a slightly different context. However, it is customary to refer to the martingale measure obtained by letting θ^2 equal the zero-vector as the minimal martingale measure. Note also that in the

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degenerate case where the market is complete, that is, $\|\tilde{\Sigma}_{\pi}\| = 0$, the minimal martingale measure price equals the unique price of the derivative that allows for a perfect replication of the derivative payoff.

It is interesting to note that had we defined the no-good-deal price based on a direct constraint on the instantaneous Sharpe ratio we would have had little hope in computing the corresponding upper and lower bounds of the derivative. Hence, the fact that we impose a bound on the market price of risk process rather than on the instantaneous Sharpe ratio itself is more of a technical nature. After all, it is the instantaneous Sharpe ratio of the derivative that we want to control. We further note that if we can find trading strategies such that the upper bounds of the instantaneous squared Sharpe ratios are attainable then we can quantify the precise contribution of adding the derivative to the opportunity set. This is the topic of Section 5.

5 | KELLY INDIFFERENCE PRICING

In this section we study the investment strategies of a Kelly trader. We follow Bermin and Holm (2019) and define a Kelly strategy as a trading strategy with maximal instantaneous Sharpe ratio. We show that a Kelly trader can never reduce the instantaneous Sharpe ratio by adding a derivative to his existing portfolio. Moreover, if the instantaneous Sharpe ratio of the derivative meets a precise condition the Kelly trader is indifferent whether to trade the derivative or not. This induces a unique price for the derivative, somewhat similar in spirit to the utility indifference pricing in Davis (1997).

Theorem 5.1 (Kelly). Any \mathbb{F} -predictable trading strategy w, with $w_{N+1} = 0$, that maximizes the magnitude of the instantaneous Sharpe ratio is of the form

$$w(t) = k(t)w^{*}(t), \quad w^{*}(t) = \sigma_{\text{diag}}^{-1}(t)\rho^{-1}(t)s(t) = \arg\max_{w(t)} \mu_{X}(t)$$

for some real-valued \mathbb{F} -predictable process k. We call such strategies Kelly strategies and we refer to the process k as the Kelly multiplier. The instantaneous squared Sharpe ratio of a Kelly strategy is independent of k and equals

$$s_X^2(t) = s'(t)\rho^{-1}(t)s(t).$$

The corresponding drift and volatility of such a trading strategy satisfy

$$\mu_X(t) = r(t) + \frac{1}{2}k(t)(2 - k(t))s_X^2(t), \quad \sigma_X^2(t) = k^2(t)s_X^2(t),$$

such that μ_X is maximal for k = 1. We call w^* the growth optimal Kelly strategy.

Proof. Below we sketch the proof as provided in Bermin and Holm (2019). First use Equation (1) to express $w^* = V^{-1}\sigma_{\text{diag}}s$. Second, define the inner product $\langle u, v \rangle_V = u'Vv$ such that

$$s_X(t) = \frac{w'(t)\sigma_{\text{diag}}(t)s(t)}{\sigma_X(t)} = \frac{\langle w(t), w^*(t) \rangle_{V(t)}}{\sqrt{\langle w(t), w(t) \rangle_{V(t)}}}.$$

Third, apply the Cauchy–Schwartz inequality to conclude that $s_X^2 \leq \langle w^*, w^* \rangle_V$, with equality if and only if w and w^* are collinear. Finally, apply the first-order condition to the general expression

$$\mu_X(t) = r(t) + \langle w(t), w^*(t) \rangle_{V(t)} - \frac{1}{2} \langle w(t), w(t) \rangle_{V(t)},$$

and verify that $\arg \max \mu_X = w^*$. The proof concludes by calculating the local characteristics for trading strategies collinear to w^* , that^w is, trading strategies of the form $w = kw^*$.

To explain the properties of a Kelly strategy we first observe that, for any investment horizon, the expected growth rate

$$\mathbb{E}\left[\frac{1}{T}\log\frac{X(T)}{X(0)}\right] = \frac{1}{T}\int_0^T \mathbb{E}[\mu_X(t)]\,dt$$

is maximal for the growth optimal Kelly strategy (also known as the Kelly criterion). This strategy seeks to optimize the growth rate by emphasizing both on directional and volatility trading. However, as pointed out already in Samuelson (1971), see also Thorp (2011), the growth optimal Kelly strategy is a highly risky investment. The standard approach to mitigate the risk has been to identify the logarithmic return with a logarithmic utility function and thereafter replace this utility function with a power utility. It was shown in MacLean et al. (1992) that under the assumption of lognormality (i.e., constant model parameters for the primary assets) the trading strategy maximizing the expected terminal power utility, with negative power, takes the form presented in Theorem 5.1, with a Kelly multiplier in the interval [0, 1]. The authors called these trading strategies fractional Kelly strategies. In Davis and Lleo (2013) it was shown that when the lognormality assumption is relaxed it is still possible to interpret the fractional Kelly strategies as those strategies maximizing the expected terminal power utility, provided that we replace the bank account with a mutual fund closely related to the intertemporal hedge portfolio in Merton (1973a). The approach taken in Bermin and Holm (2019), however, is different in the sense that they argue that the fundamental property of a Kelly strategy is that it has a maximal instantaneous Sharpe ratio. For such trading strategies it follows that the Kelly multiplier should lie in the interval [0, 1] as otherwise the same instantaneous logarithmic return can always be achieved at a lower volatility. While the Sharpe ratio is an important quantity for short-term horizons (in particular when dynamically rebalancing a trading strategy) it is less so for longer investment horizons. The reason is that, overtime, the portfolio distribution is often skewed as prominent traders tend to have lots of gains and few losses. Hence, an additional risk metric is typically needed. We argue that the Kelly multiplier effectively controls the risk intrinsic to the portfolio (notably the drawdown risk) in a more transparent way than any parametric utility representation. As a general guideline we propose a value of k around 1/2 to reduce the risk of the growth optimal Kelly strategy. Before we can extend Theorem 5.1, to include the derivative asset, we first present a supplementary technical result.

Lemma 5.2. Suppose that the matrix

$$\hat{\rho}(t) = \begin{pmatrix} \rho(t) & \rho_{\pi}(t) \\ \rho_{\pi}'(t) & 1 \end{pmatrix}$$

is a.s. positive definite. Then

$$\hat{\rho}^{-1}(t) = \frac{1}{h(t)} \begin{pmatrix} h(t)\rho^{-1}(t) + \rho^{-1}(t)\rho_{\pi}(t)\rho_{\pi}'(t)\rho^{-1}(t) & -\rho^{-1}(t)\rho_{\pi}(t) \\ -\rho_{\pi}'(t)\rho^{-1}(t) & 1 \end{pmatrix}$$

where the real-valued \mathbb{F} -adapted process $h = 1 - \rho'_{\pi} \rho^{-1} \rho_{\pi} \in (0, 1]$ a.s.

Proof. Since $\hat{\rho}$ is a.s. positive definite the inverse $\hat{\rho}^{-1}$ exists and is also a.s. positive definite. Furthermore, ρ is a.s. positive definite since every principal submatrix of a positive definite matrix is positive definite. Therefore ρ^{-1} is also a.s. positive definite, which yields $1 - h = \rho'_{\pi} \rho^{-1} \rho_{\pi} \ge 0$. By using the rules for block matrix determinants we further have det $\hat{\rho} = h \det \rho$. Finally, since the determinant of a positive definite matrix is strictly positive it follows that h > 0. The particular form of $\hat{\rho}^{-1}$ is easily verified by a direct calculation.

We are now ready to present the details about the Kelly strategy when the derivative is included in the opportunity set. We summarize the results below.

Proposition 5.3. Let k be a real-valued \mathbb{F} -predictable process. The instantaneous Sharpe ratio corresponding to the primary asset trading strategy

$$w(t) = k(t)\sigma_{\text{diag}}^{-1}(t)\rho^{-1}(t)s(t)$$

satisfies

$$s_X^2(t) = s'(t)\rho^{-1}(t)s(t).$$

Furthermore, the trading strategy

$$w(t) = k(t)\sigma_{\text{diag}}^{-1}(t)\rho^{-1}(t) \left(s(t) - \frac{s_{\pi}(t) - \rho_{\pi}'(t)\rho^{-1}(t)s(t)}{1 - \rho_{\pi}'(t)\rho^{-1}(t)\rho_{\pi}(t)}\rho_{\pi}(t) \right), w_{N+1}(t) = k(t)\sigma_{\pi}^{-1}(t)\frac{s_{\pi}(t) - \rho_{\pi}'(t)\rho^{-1}(t)s(t)}{1 - \rho_{\pi}'(t)\rho^{-1}(t)\rho_{\pi}(t)},$$

taking positions in both the primary assets and the derivative has an instantaneous Sharpe ratio that satisfies

$$s_X^2(t) = s'(t)\rho^{-1}(t)s(t) + \frac{\left(s_\pi(t) - \rho'_\pi(t)\rho^{-1}(t)s(t)\right)^2}{1 - \rho'_\pi(t)\rho^{-1}(t)\rho_\pi(t)}, \ge s'(t)\rho^{-1}(t)s(t).$$

Finally, we note that

$$\sigma_{\pi}^{2}(t) \Big(1 - \rho_{\pi}'(t) \rho^{-1}(t) \rho_{\pi}(t) \Big) = \| \widetilde{\Sigma}_{\pi}(t) \|^{2},$$

$$\sigma_{\pi}(t) \rho_{\pi}'(t) \rho^{-1}(t) s(t) = \Sigma_{\pi}'(t) \theta^{1}(t).$$

Proof. The first result for the primary asset Kelly strategy follows from Theorem 5.1. For the second result, we define $\hat{w} = (w', w_{N+1})'$, $\hat{s} = (s', s_{\pi})'$, and $\hat{\sigma}_{\text{diag}} = \text{diag}((\sigma', \sigma_{\pi})')$, such that the extended Kelly strategy takes the form $\hat{w} = k\hat{\sigma}_{\text{diag}}^{-1}\hat{\rho}^{-1}\hat{s}$, with an associated instantaneous Sharpe ratio given by $s_X^2 = \hat{s}'\hat{\rho}^{-1}\hat{s}$. The expressions now follow from straightforward calculations and Lemma 5.2. Note that the last inequality follows from the bounds previously derived for the process *h*. For the final results, we first compare the elements of the inverse of the extended asset–asset covariance matrix

$$\hat{V}^{-1}(t) = \left(\begin{pmatrix} \Sigma(t) & \mathbf{0}_{N \times M - N} \\ \Sigma_{\pi}'(t) & \widetilde{\Sigma}_{\pi}'(t) \end{pmatrix} \begin{pmatrix} \Sigma'(t) & \Sigma_{\pi}(t) \\ \mathbf{0}_{M - N \times N} & \widetilde{\Sigma}_{\pi}(t) \end{pmatrix} \right)^{-1} = \begin{pmatrix} \Sigma(t)\Sigma'(t) & \Sigma(t)\Sigma_{\pi}(t) \\ (\Sigma(t)\Sigma_{\pi}(t))' & \sigma_{\pi}^{2}(t) \end{pmatrix}^{-1}$$

with those of the alternative expression $\hat{V}^{-1} = \hat{\sigma}_{\text{diag}}^{-1} \hat{\sigma}_{\text{diag}}^{-1}$. This shows that $\sigma_{\pi}^2 \left(1 - \rho_{\pi}' \rho^{-1} \rho_{\pi}\right) = \|\tilde{\Sigma}_{\pi}\|^2$. Thereafter, we use Equation (5) to derive the last identity.

By comparing Proposition 5.3 with Proposition 4.1 we see that we have found the trading strategies that attain the Hansen–Jagannathan bounds. It should come as no surprise that the bounds are attained by Kelly strategies since these are maximal instantaneous Sharpe ratio strategies. The practical relevance is illustrated in the example below.

Example 5.4. Consider a Kelly strategy w, derivative excluded, with Kelly multiplier $k \in [0, 1]$ and instantaneous Sharpe ratio s_X . Let \hat{w} be another Kelly strategy, derivative included, with Kelly multiplier \hat{k} and instantaneous Sharpe ratio \hat{s}_X . By setting $\hat{k} = ks_X/\hat{s}_X$ it follows from Theorem 5.1 and Proposition 5.3 that $\hat{\sigma}_X = \sigma_X$ and $\hat{\mu}_X = \mu_X + \sigma_X(\hat{s}_X - s_X) \ge \mu_X$.

We see that a Kelly trader can always increase the instantaneous return, for a fixed volatility, if he can increase the instantaneous Sharpe ratio. It also follows from Proposition 5.3 that if the market is complete, that is, $\|\tilde{\Sigma}_{\pi}\| = 0$, the instantaneous Sharpe ratio of a portfolio taking positions in both the primary assets and the derivative explodes (giving rise to arbitrage opportunities) unless $s_{\pi} = \rho'_{\pi} \rho^{-1} s$. This observation leads us to the concept of indifference pricing.

Definition 5.5. We say that an investor is indifferent to trade a derivative if the allocation in the primary assets is unaffected by adding the derivative to the opportunity set and the additional weight allocated to the derivative equals zero.

Theorem 5.6. A Kelly trader is indifferent to trade a derivative if and only if

$$s_{\pi}(t) = \rho'_{\pi}(t)\rho^{-1}(t)s(t) = \frac{1}{\sigma_{\pi}(t)}\Sigma'_{\pi}(t)\theta^{1}(t), \quad 0 \le t \le T.$$

Proof. The proof follows directly from Proposition 5.3.

It is quite remarkable that every Kelly trader, characterized by his attitude towards risk in terms of the Kelly multiplier, agrees on the indifference price. This observation generally contrasts the similar approach based on utility functions. In the following results we show how to translate the precise condition on the instantaneous Sharpe ratio of the derivative to pricing by means of no-arbitrage.

Theorem 5.7. A Kelly trader is indifferent to trade a derivative if and only if every no-good-deal price $\{\Pi_c\}_{c\geq 0}$ is given by Π_0 , that is, the unique minimal martingale measure price.

Proof. Without loss of generality we assume that $\|\tilde{\Sigma}_{\pi}\| > 0$ since otherwise the market collapses to a complete market. It is clear, from Definition 4.3, that if $\Pi_c = \Pi_0$, for all $c \ge 0$, then θ^2 is the zero-vector. Theorem 3.5 then states that both θ_{\perp}^2 and θ_{\parallel}^2 are zero-vectors, which implies that $\sigma_{\pi} s_{\pi} = \sum_{\pi} \theta^1$. Hence, according to Theorem 5.6, a Kelly trader is indifferent to trade the derivative.

For the other direction we assume that $\sigma_{\pi} s_{\pi} = \sum_{\pi}^{\prime} \theta^{1}$. Theorem 3.5 then states that the market price of risk vector θ^{2} satisfies

$$\theta^{2}(t) = \left(I - \|\widetilde{\Sigma}_{\pi}(t)\|^{-2}\widetilde{\Sigma}_{\pi}(t)\widetilde{\Sigma}_{\pi}'(t)\right)\phi(t),$$

for some arbitrary \mathbb{F} -adapted process ϕ taking values in \mathbb{R}^{M-N} . This implies that $\Sigma \theta^2 = 0$ a.s. for all $t \leq T$. Hence, as proved in Lemma A1, it now follows that the no-good-deal price, for each fixed $c \geq 0$, is unique and equal to Π_0 .

Corollary 5.8. A Kelly trader is indifferent to trade the derivative if and only if every no-arbitrage price is given by the unique minimal martingale measure price.

Proof. The proof follows since the no-good-deal price interval, with index *c*, approaches the no-arbitrage price interval as $c \rightarrow \hat{c}$, according to Equation (7).

We have shown that a Kelly trader can associate a unique no-arbitrage price to a derivative. This does not mean, however, that the market must trade the derivative at said price. It only tells us that a Kelly trader can increase the magnitude of his instantaneous Sharpe ratio (alternatively, as shown in Example 5.4, increase his instantaneous logarithmic return at no additional volatility) should the derivative not be traded at this price. We proceed by providing further evidence for why the market will price a derivative according to the Kelly indifference price; or equally the minimal martingale measure price.

6 | KELLY EQUILIBRIUM PRICING

If we consider the market as consisting of a number of investors we can expect their joint actions of trading to force the market to some kind of trading equilibrium. In this section we make precise what such a trading equilibrium means for a derivative product. We assume that each investor tries to maximize the magnitude of the instantaneous Sharpe ratio corresponding to his portfolio. This allows us to formulate the equilibrium as the solution to a reduced two-person zero-sum differential game and apply the min–max concept of von Neumann (1928). It is well known that any game theory equilibrium found in this way can also be seen as a Nash equilibrium.

Let us consider a Kelly trader, applying an arbitrary Kelly multiplier k, who can invest in the primary assets and the derivative. If we denote the trading strategy by $\hat{w} = (w', w_{N+1})'$, then, as shown in Corollary 4.2 and Proposition 5.3, we have

$$\min_{\mu_{\pi}(t)} \max_{\hat{w}(t)} s_X^2(t) = s'(t) \rho^{-1}(t) s(t),$$

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where the growth optimal trading strategy \hat{w}^* is defined as in Proposition 5.3 under the additional constraint that μ_{π}^* is set such that $s_{\pi} = \rho_{\pi}' \rho^{-1} s$. This choice of μ_{π}^* implies that the Kelly indifference price equals the minimal martingale measure price and that the Kelly trader takes no positions in the derivative since $w_{N+1}^* = 0$.

Rather than looking at the interactions between every market investor we take the approach that all the other investors gang up on our Kelly trader. Hence, the other traders (not necessarily Kelly traders) will try to minimize the magnitude of the instantaneous Sharpe ratio for every trading strategy being used. The point of equilibrium can be described as below.

Theorem 6.1. Given a Kelly trader who can invest in the primary assets and the derivative. Then

$$\min_{\mu_{\pi}(t)} \max_{\hat{w}(t)} s_X^2(t) = \max_{\hat{w}(t)} \min_{\mu_{\pi}(t)} s_X^2(t) = s'(t)\rho^{-1}(t)s(t).$$

The equilibrium solution (μ_{π}^*, \hat{w}^*) is given by

$$\mu_{\pi}^{*}(t) = r(t) + \sigma_{\pi}(t)\rho'_{\pi}\rho^{-1}(t)s(t) - \frac{1}{2}\sigma_{\pi}^{2},$$

w*(t) = k(t)\sigma_{\text{diag}}^{-1}(t)\rho^{-1}(t)s(t), w_{N+1}^{*}(t) = 0.

Proof. Since we have already shown the market coalition's response to a Kelly trader it remains to consider the response of a Kelly trader to the market coalition. The first-order condition corresponding to the optimization problem $\min_{\mu_{\pi}} s_X^2$ is $w_{N+1} = 0$. A Kelly trader facing such a constraint will trade in the primary assets using the strategy $w = k\sigma_{\text{diag}}^{-1}\rho^{-1}s$. Moreover, to enforce that $w_{N+1} = 0$ the market coalition will set μ_{π} such that $s_{\pi} = \rho'_{\pi}\rho^{-1}s$. This concludes the proof.

We have shown that the market is in equilibrium, in the sense described above, if and only if a Kelly trader is indifferent to trade the derivative. This is a very natural condition since any trader can either choose to be long or short the derivative and for each long/short position there is an opposite short/long counterpart. Hence, a derivative price resulting in a Kelly trader being indifferent to trade the derivative can indeed be seen as a fair price. Note also that if all the market participants are Kelly traders the market price must be in equilibrium since each contract requires a buyer and a seller. By the use of Theorem 5.7 we now state that the market is in equilibrium if and only if every no-good-deal price is given by the unique minimal martingale measure price and the statement extends to the entire arbitrage-free price interval by the use of Corollary 5.8.

It is illustrative to analyze the situation where the market does not value the derivative according to the equilibrium price Π_0 . As explained in Example 5.4 a Kelly trader can then increase his return, at no additional volatility, by adding the derivative to his portfolio. Moreover, from Equation (4) and Theorem 6.1 we see that

$$d\log\frac{\Pi(t)}{\Pi_0(t)} = \sigma_{\pi}(t) \Big(s_{\pi}(t) - s_{\pi}^*(t) \Big) dt, \quad s_{\pi}^*(t) = \rho_{\pi}'(t) \rho^{-1}(t) s(t),$$

since the volatility of the derivative is invariant for all no-arbitrage prices. This yields

$$\log \frac{\Pi(t)}{\Pi_0(t)} = -\int_t^T \sigma_\pi(u)(s_\pi(u) - s_\pi^*(u)) \, du, \quad 0 \le t \le T.$$

Heuristically, we now let *t* run backwards from *T* to zero and identify the first time $\tau > 0$ such that $\Pi(T - \tau) \neq \Pi_0(T - \tau)$. Then $s_\pi \ge s_\pi^*$ if $\Pi \le \Pi_0$, which implies that $w_{N+1} \ge 0$, according to Proposition 5.3. If we further assume, as in Bermin and Holm (2019), that buying (selling) an asset has a positive (negative) impact on the price of the asset, it is clear that every Kelly trader will push the market price of the derivative Π towards the equilibrium price Π_0 , should they differ. Under the theoretical assumption that the adjustment takes place instantaneously we can repeat the argument; letting *t* run backwards from τ to zero and so on. Hence, without diving into the technical details we argue that the equilibrium price is dynamically stable if the Kelly traders dominate the market.

Let us end this section by describing how a Kelly trader goes about assessing a price to a derivative. For the sake of simplicity we focus on a growth optimal Kelly trader employing a Kelly multiplier k = 1. If the growth optimal Kelly trader is only allowed to trade in the *N* primary assets we know from Proposition 5.3 that the trading strategy equals

$$w(t) = \sigma_{\text{diag}}^{-1}(t)\rho^{-1}(t)s(t).$$

The Kelly trader then adds M - N derivatives to his opportunity set

$$d\log \Pi_m(t) = \mu_{\pi|m}(t)dt + \Sigma'_{\pi|m}(t)dW^1(t) + \widetilde{\Sigma}'_{\pi|m}(t)dW^2(t)$$

in such a way that the extended correlation matrix

$$\hat{\rho}(t) = \begin{pmatrix} \rho(t) & \rho_{\pi}(t) \\ \rho_{\pi}'(t) & \tilde{\rho}_{\pi}(t) \end{pmatrix}$$

is invertible. Here, the blocks are defined, for each $n \in \{1, ..., N\}$, according to

$$\begin{split} \rho_{\pi|n,m}(t) &= \frac{\Sigma'_{\pi|m}(t)\Sigma_n(t)}{\sigma_{\pi|m}(t)\sigma_n(t)}, \quad 1 \le m \le M - N, \\ \sigma_{\pi|m}^2(t) &= \|\Sigma_{\pi|m}(t)\|^2 + \|\widetilde{\Sigma}_{\pi|m}(t)\|^2, \quad 1 \le m \le M - N, \\ \widetilde{\rho}_{\pi|u,v}(t) &= \frac{\Sigma'_{\pi|u}(t)\Sigma_{\pi|v}(t) + \widetilde{\Sigma}'_{\pi|u}(t)\widetilde{\Sigma}_{\pi|v}(t)}{\sigma_{\pi|u}(t)\sigma_{\pi|v}(t)}, \quad 1 \le u, v \le M - N \end{split}$$

If we further let $\hat{\sigma}_{\text{diag}} = \text{diag}((\sigma', \sigma_{\pi|1}, ..., \sigma_{\pi|M-N})')$, then

$$\begin{pmatrix} w(t) \\ \mathbf{0}_{M-N} \end{pmatrix} = \hat{\sigma}_{\text{diag}}^{-1}(t) \hat{\rho}^{-1}(t) \begin{pmatrix} s(t) \\ \rho'_{\pi}(t) \rho^{-1}(t) s(t) \end{pmatrix},$$
(8)

since elementary matrix manipulations yield

$$\binom{s(t)}{\rho'_{\pi}(t)\rho^{-1}(t)s(t)} = \hat{\rho}(t) \binom{\rho^{-1}(t)s(t)}{\mathbf{0}_{M-N}}$$

We now consider a growth optimal Kelly strategy which takes positions in both the primary assets and in the derivatives. Such a strategy can be represented by $\hat{w} = \hat{\sigma}_{\text{diag}}^{-1}\hat{\rho}^{-1}\hat{s}$, where $\hat{s} = (s', s'_{\pi})'$ and $s_{\pi} = (s_{\pi|1}, ..., s_{\pi|M-N})'$ denote the instantaneous Sharpe ratio of the derivatives. A simple comparison with Equation (8) then shows that the indifference price corresponding to each of the added derivatives satisfies the vector equation

$$s_{\pi}(t) = \rho'_{\pi}(t)\rho^{-1}(t)s(t).$$
(9)

Since the trading strategy of any Kelly trader is proportional to that of a growth optimal Kelly trader (via the Kelly multiplier) it follows that every Kelly trader agrees on the indifference prices. Note further that the condition related to the indifference price, Equation (9), is independent of whether the market is complete or incomplete. Should the market price of a derivative not equal the indifference price a Kelly trader will add the derivative to his portfolio and thereby increase his instantaneous Sharpe ratio. The contribution of this paper is to identify the Kelly indifference price with the unique minimal martingale measure price, as characterized in Definition 3.8, with $\theta^1 = \Sigma^{-1} \sigma_{\text{diag}} s$ and θ^2 equal to the zero-vector. This further implies that under the additional assumption of a complete market, in which case the matrices $\tilde{\Gamma}$ and $\tilde{\Sigma}$ vanish, the Kelly indifference price agrees with the unique no-arbitrage price of Merton (1973b).

7 | DISCUSSIONS

In this section we provide some general remarks about Kelly trading. We also discuss limitations and possible extensions to the results presented earlier. While we have striven to keep the model specifications as general as possible, we do assume that the source of randomness is generated by a Brownian motion. It would be interesting to investigate our approach when random jumps are included.

Regarding the notion of arbitrage employed in this paper, we have deliberately used the standard definition. Reason being that it enables us to easily prove the existence of a market price of risk process, without having to impose

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conditions on the underlying model parameters. For particular model choices, however, one always has to check whether, say, calendar spreads are priced in compliance with the absence of arbitrage.

For many people the notion of Kelly trading is either not well understood or not fully accepted. To be precise, the growth optimal Kelly strategy is well understood but not fully accepted, while the so-called fractional Kelly strategies in MacLean et al. (1992) and Davis and Lleo (2013) are (somewhat) well accepted but not fully understood. Over the years there has been a fierce debate between the two camps of expected terminal utility maximizers and Kelly promoters, with Samuelson in particular taking a strong side against, notably, growth optimal Kelly trading. A summary of his objections can be found in a detailed response by Ziemba (2014) and for clarity we repeat the three main claims here. First, Samuelson argued that the growth optimal Kelly strategy is a very risky investment, especially in the short term. Second, he argued that Kelly trading is not consistent with maximization of expected terminal utility for utility functions other than log. Third, he indicated that Kelly trading results in highly concentrated, not diversified, portfolio allocations. It is our view that the fractional Kelly strategies essentially solve the first two objections but at the expense of either generality or economic intuition. Let us recap the situation: MacLean et al. (1992) proved that Kelly trading is consistent with power utility functions when the model parameters of the primary assets are constant, while Davis and Lleo (2013) showed that the restriction on the model parameters can be relaxed if we replace the bank account with a particular, theoretically constructed, mutual fund. Hence, while none of these approaches is fully satisfactory, we believe that the definition laid out in Bermin and Holm (2019), where Kelly strategies are simply identified as those strategies with maximal instantaneous Sharpe ratio, is better suited for the wider audience. For those who, nonetheless, wonder what the corresponding utility interpretation of an instantaneous Sharpe maximal strategy looks like, we recall (using Itô's lemma) that

$$\mathbb{E}[u(X(T))|\mathcal{F}(t)] = u(X(t)) + \int_t^T \mathbb{E}\left[u'(X(s))X(s)\left(\mu_X(s) + \frac{1}{2}(1 - p(X(s)))\sigma_X^2(s)\right)\right] \mathcal{F}(t)\right] ds,$$

with *p* denoting the relative risk aversion of Arrow–Pratt. Therefore

$$\frac{1}{p(X(t))}w^*(t) = \arg\max_{w(t)} \frac{d}{dT} \mathbb{E}[u(X(T))|\mathcal{F}(t)]\Big|_{T=t}$$

which tells us that maximizing the terminal utility over an infinitesimal investment horizon can always be characterized by an instantaneous Sharpe maximal Kelly strategy when the Kelly multiplier is set to the reciprocal of the relative risk aversion. It is unlikely, though, that Kelly traders consider themselves to be, say, rolling one day expected utility maximizers. Reason being that if one strongly believes that financial risk is adequately captured by a fixed utility function, over a finite investment horizon, the random fluctuations of the model parameters should be hedged as demonstrated in Merton (1973a). Instead, Kelly traders most likely set the Kelly multiplier based on some other global risk criteria, for instance, drawdown risk. With that being said, it is worth mentioning that there is strong evidence that an individual's aversion to risk is not a characteristic set in stone, see Guiso et al. (2018). Risk aversion can change, both due to age but also due to external factors, such as a financial crisis. While it is phenomenally difficult to associate a utility function with these observations the Kelly framework faces no such problems. Here, the relative risk aversion is simply implied from the Kelly multiplier which, at a given time, accounts for an investor's willingness to leverage. Finally, let us comment on Samuelson's third point, namely, concentration risk. Clearly a Kelly strategy can generate highly nondiversified allocations. As an example we consider the case where all but one of the primary assets are replaced by a mutual fund that has close to maximal instantaneous Sharpe ratio. A Kelly strategy applied to the mutual fund and the remaining primary asset will in principle allocate all the resources to the mutual fund; as it should. For more normal situations, however, it is our view that the Kelly approach, per se, does not result in overly nondiversified allocations. Rather it is the use of poor historical estimates of the model parameters that may cause this outcome. It is well known that, although the covariances of the primary assets can be adequately estimated, it is genuinely difficult to estimate the expected logarithmic returns. It would be interesting to study the impact of estimation errors in greater details.

To shed further light on Kelly strategies we additionally consider the larger group of alpha-seeking traders, where alpha refers to Jensen's alpha as introduced in Jensen (1964). We let

$$b_{\pi}(t) = \mu_{\pi}(t) + \frac{1}{2}\sigma_{\pi}^{2}(t), \quad b_{X}(t) = \mu_{X}(t) + \frac{1}{2}\sigma_{X}^{2}(t)$$

denote the instantaneous return of a derivative and a portfolio X, respectively, such that the instantaneous alpha of a derivative measured against a benchmark portfolio X takes the form

$$\begin{aligned} \alpha_{\pi,X}(t) &= b_{\pi}(t) - r(t) - \beta_{\pi,X}(t)(b_{X}(t) - r(t)) \\ &= \sigma_{\pi}(t)s_{\pi}(t) - \frac{\sigma_{\pi}(t)\rho_{\pi,X}(t)}{\sigma_{X}(t)}\sigma_{X}(t)s_{X}(t) \\ &= \sigma_{\pi}(t)(s_{\pi}(t) - \rho_{\pi,X}(t)s_{X}(t)). \end{aligned}$$

Consequently, we may say that an alpha trader is indifferent to trade a derivative if $s_{\pi} = \rho_{\pi,X} s_X$. Straightforward calculations show that when *X* is a Kelly portfolio this condition boils down to Theorem 5.6. Moreover, if $\alpha_{\pi,X}$ is different from zero it is well known, see Nielsen and Vassalou (2004), that the instantaneous Sharpe ratio of portfolio *X* can be improved by allocating a tiny fraction of the wealth to either a long or a short position in the derivative. Hence, in some sense, we can view Kelly traders as optimal alpha traders, providing precise instructions how to reach the Sharpe maximal trading strategy.

It is further interesting to note that while arbitrageurs can only establish upper and lower bounds for the price of a derivative in an incomplete market, Kelly traders consider the derivative price to be unique. Hence, by observing how far the market derivative price is from the Kelly indifference price we can, in a way, infer the composition of the market participants. For instance, it would be interesting to study the composition in various single name (or index) option markets, assuming those are priced with stochastic volatility models of the form

$$d\log P_1(t) = \left(r(t) - \frac{1}{2}\sigma_1^2(t)\right)dt + \sigma_1(t)dV_{\theta}^1(t),$$

$$d\sigma_1(t) = g(t)dt + h(t)\left(\rho_{\sigma,P}(t)dV_{\theta}^1(t) + \sqrt{1 - \rho_{\sigma,P}^2(t)}dV_{\theta}^2(t)\right)$$

for some \mathbb{F} -adapted processes g and h. If Kelly traders were dominant among the market participants we would see that $g \approx \gamma - h\rho_{\sigma,P}s_1$, with γ being the drift of σ_1 under the historical measure as stated in Definition 3.8. Alternatively, we could in this case claim that a stochastic volatility model $(g, h, \rho_{\sigma,P})$ is consistent with the underlying asset if the instantaneous Sharpe ratio $s_1 \approx (\gamma - g)/(h\rho_{\sigma,P})$. Such analysis, though, goes beyond the scope of this paper.

We end this section with some additional details related to the choice between trading a single asset and/or trading an option written on the asset. That is, we consider the case where N = 1, such that $\rho = 1$ and $\rho_{\pi}^2 \leq 1$. Then, as shown in Proposition 5.3, the asset-derivative instantaneous correlation $\rho_{\pi} = \pm 1$ (with the sign depending on whether we consider a call or a put option) if and only if the volatility component of the derivative not spanned by the asset, $\|\tilde{\Sigma}_{\pi}\|$, equals zero. Hence, in an arbitrage-free complete market the instantaneous squared Sharpe ratio of the derivative equals that of the asset, meaning that a Sharpe-maximizing Kelly trader is indifferent as to which one to trade. Next, let us consider an incomplete market (e.g., the stochastic volatility model presented previously) where $\|\tilde{\Sigma}_{\pi}\| > 0$, such that $\rho_{\pi}^2 < 1$. Two situations can now occur. If the market price of the derivative does not agree with the Kelly indifference price a Kelly trader can increase his instantaneous Sharpe ratio (alternatively, as shown in Example 5.4, increase his instantaneous return at no additional volatility) by trading both the asset and the derivative. If, however, the market price of the derivative agrees with the Kelly indifference price then, as shown in Theorem 5.6, a Kelly trader is indifferent whether to add the derivative to an existing portfolio having positions in the asset. Furthermore, if the Kelly trader must choose to trade in either the asset or the derivative he will, in this case, always pick the asset since $s_{\pi}^2 = \rho_{\pi}^2 s^2 < s^2$.

8 | CONCLUSIONS

In this paper we show that if the capital market is in equilibrium, in the sense that a Kelly trader can do no better (in terms of maximizing the magnitude of his instantaneous Sharpe ratio) by trading the derivative given the actions of the other market participants, the no-arbitrage price of the derivative must equal that of the minimal martingale measure price in Föllmer and Schweizer (1991). The proof relies on showing that a Kelly trader is indifferent to trade the derivative if and only if every no-good-deal price in Cochrane and Saá Requejo (2000) is unique. Since the maximal no-good-deal price interval and a Kelly trader is indifferent to trade the derivative when the market is in a trading equilibrium the result follows. Hence, by adding elements from the optimal growth theory of Kelly (1956) and Latané (1959) we derive a unique indifference and equilibrium price for derivatives in incomplete markets.

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DATA AVAILABILITY STATEMENT

Data sharing not applicable to this article as no data sets were generated or analyzed during the current study.

ORCID

Hans-Peter Bermin D http://orcid.org/0000-0003-1665-6832

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APPENDIX A

In this appendix we state the Hamilton-Jacobi-Bellman equation associated with the upper and lower no-good-deal prices. Without loss of generality we can view the nontradable indices as factors such that the model parameters allow a Markovian representation with respect to (P, I). This means, for instance, that there exists a function $r: \mathbb{R}_+ \times \mathbb{R}^N_+ \times \mathbb{R}^{M-N} \to \mathbb{R}$ such that

$$r(t) = r(t, P(t), I(t)),$$

and similar for the other terms. Additionally, we also assume that the market price of risk θ also allows for a Markovian representation. Let further $F: [0, T] \times \mathbb{R}^N_+ \times \mathbb{R}^{M-N} \to \mathbb{R}_+$ be a deterministic function, such that the infinitesimal operator \mathcal{A}^{θ} of (P, I), with respect to the probability measure \mathbb{P}_{θ} , takes the form

$$\begin{aligned} \mathcal{A}^{\theta}F(t,x,y) &= \sum_{i=1}^{N} \frac{\partial F}{\partial x_{i}}(t,x,y)x_{i}r(t,x,y) + \sum_{j=1}^{M-N} \frac{\partial F}{\partial y_{j}}(t,x,y)\gamma_{j}(t,x,y) \\ &- \sum_{j=1}^{M-N} \frac{\partial F}{\partial y_{j}}(t,x,y)(\Gamma_{j}'(t,x,y)\theta^{1}(t,x,y) + \widetilde{\Gamma}_{j}'(t,x,y)\theta^{2}(t,x,y)) \\ &+ \frac{1}{2}\sum_{i,k=1}^{N} \frac{\partial^{2}F}{\partial x_{i}\partial x_{k}}(t,x,y)x_{i}x_{k}\Sigma_{i}'(t,x,y)\Sigma_{k}(t,x,y) \\ &+ \sum_{i=1}^{N}\sum_{j=1}^{M-N} \frac{\partial^{2}F}{\partial x_{i}\partial y_{j}}(t,x,y)x_{i}\Sigma_{i}'(t,x,y)\Gamma_{j}(t,x,y) \\ &+ \frac{1}{2}\sum_{j,k=1}^{M-N} \frac{\partial^{2}F}{\partial y_{j}\partial y_{k}}(t,x,y)\Gamma_{j}'(t,x,y)\Gamma_{k}(t,x,y) \\ &+ \frac{1}{2}\sum_{i,k=1}^{M-N} \frac{\partial^{2}F}{\partial y_{j}\partial y_{k}}(t,x,y)\widetilde{\Gamma}_{j}'(t,x,y). \end{aligned}$$

Given $c \in \mathbb{R}_+$, we now specify the function

$$F_c(t, x, y) = B(t) \underset{\|\theta^2\| \le c}{\mathbb{E}}_{\theta} \left[\frac{\Phi(P(T))}{B(T)} | P(t) = x, I(t) = y \right], \quad \theta \in \Theta_T^{\mathcal{M}},$$

for some $\Phi : \mathbb{R}^N_+ \to \mathbb{R}_+$, such that

$$\Pi_c(t) = F_c(t, P(t), I(t)), \quad \text{a.s.} \quad 0 \le t \le T.$$

It is proven in Björk and Slinko (2006) that the no-good-deal price boundaries (min Π_c , max Π_c) can be characterized by two deterministic functions (F_c^1, F_c^2). One of these functions solves the Hamilton–Jacobi–Bellman equation:

$$\frac{\partial F_c}{\partial t}(t, x, y) + \sup_{\|\theta^2\| \le c} \mathcal{A}^{\theta} F_c(t, x, y) - r(t, x, y) F_c(t, x, y) = 0,$$
$$F_c(T, x, y) = \Phi(x)$$

while the other solves a similar equation but with sup exchanged for inf. Hence, deriving the no-good-deal boundary prices can be composed in two steps. First, compute the extremal points of

$$\sum_{j=1}^{M-N} \frac{\partial F_c}{\partial y_j}(t, x, y) \widetilde{\Gamma}'_j(t, x, y) \theta^2(t, x, y) \quad \text{s.t.} \quad \| \theta^2(t, x, y) \| \le c.$$

Thereafter, solve the partial differential equation for each solution $\hat{\theta}^2$. By introducing the gradient $\nabla_y F_c = (\frac{\partial F_c}{\partial y_1}, ..., \frac{\partial F_c}{\partial y_{M-N}})'$, it is easily seen that the static optimization problem takes the simple form

$$\theta^2(t, x, y) = \pm c \|\widetilde{\Gamma}'(t, x, y) \nabla_y F_c(t, x, y)\|^{-1} \widetilde{\Gamma}'(t, x, y) \nabla_y F_c(t, x, y),$$

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provided that there exists θ^2 such that $(\nabla_y F_c)' \tilde{\Gamma} \theta^2 \neq 0$. We further notice that every no-arbitrage price (and hence every no-good-deal price) satisfies the same partial differential equation, albeit with a particular market price of risk component θ^2 . This implies that

$$\tilde{\Sigma}_{\pi}(t,x,y) = \frac{1}{F_c(t,x,y)} \tilde{\Gamma}'(t,x,y) \nabla_y F_c(t,x,y), \quad \forall \ c \ge 0.$$

Hence, the static optimization problem can equally be stated as finding the extremal points of

 $F_c(t, x, y)\tilde{\Sigma}'_{\pi}(t, x, y)\theta^2(t, x, y) \quad \text{s.t.} \quad \|\theta^2(t, x, y)\| \le c,$

with solution $\hat{\theta}^2 = \pm c \|\tilde{\Sigma}_{\pi}\|^{-1} \tilde{\Sigma}_{\pi}$, provided that there exists θ^2 such that $\tilde{\Sigma}'_{\pi} \theta^2 \neq 0$. Below we provide a minor, but for us important, extension to the theory of no-good-deal pricing, covering the degenerate case.

Lemma A1. The no-good-deal boundary functions (F_c^1, F_c^2) collapse to F_0 , for all $c \ge 0$, if

$$\widetilde{\Sigma}'_{\pi}(t, x, y)\theta^{2}(t, x, y) = 0, \quad \forall (t, x, y) \in [0, T] \times \mathbb{R}^{N}_{+} \times \mathbb{R}^{M-N}_{+}$$

Proof. By inspecting the infinitesimal operator \mathcal{A}^{θ} , for an arbitrary function *F*, we see that

$$\sup_{\|\theta^2\|\leq c} \mathcal{A}^{\theta}F(t,x,y) = \inf_{\|\theta^2\|\leq c} \mathcal{A}^{\theta}F(t,x,y), \quad c\geq 0,$$

for every point (t, x, y) such that

$$\tilde{\Sigma}'_{\pi}(t, x, y)\theta^2(t, x, y) = 0$$

Consequently, since (F_c^1, F_c^2) satisfies the same partial differential equation and the same terminal boundary condition they must agree everywhere.